Frequency transformations in multiple dimensions

DS-GA 1013 / MATH-GA 2824 Mathematical Tools for Data Science

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Prerequisites

Calculus (complex numbers)

Linear algebra (orthogonality, basis, projections)

Fourier series

The sampling theorem

Discrete Fourier transform

Multidimensional signals

Square-integrable functions defined on a hyperrectangle
$\mathcal{I} := [a_1, b_1] \times \ldots \times [a_d, b_d] \subset \mathbb{R}^d$

Inner product:

$$\langle x, y \rangle := \int_{\mathcal{I}} x(t) \overline{y(t)} \, \mathrm{d}t.$$

Goal: Extension of frequency representations to multidimensional signals

Multidimensional sinusoid

$$a \cos \left(2\pi \langle f, t \rangle + \theta\right)$$

The frequency and time indices are now *d*-dimensional

Multidimensional sinusoid

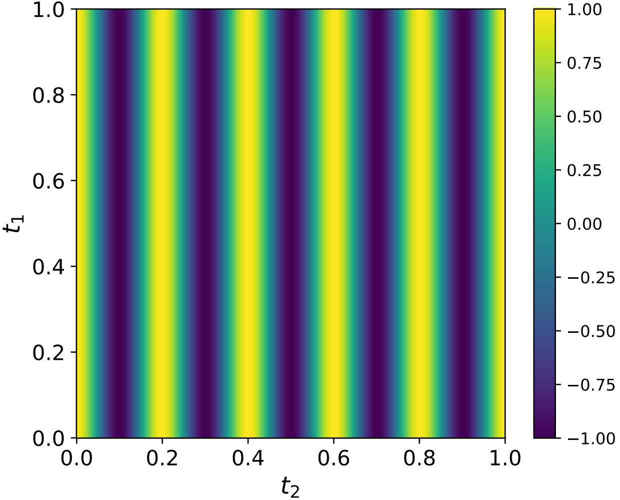
If we move in any direction v orthogonal to f

$$a \cos\left(2\pi\langle f, t + v\rangle + \theta\right) = a \cos\left(2\pi\langle f, t\rangle + \theta\right)$$

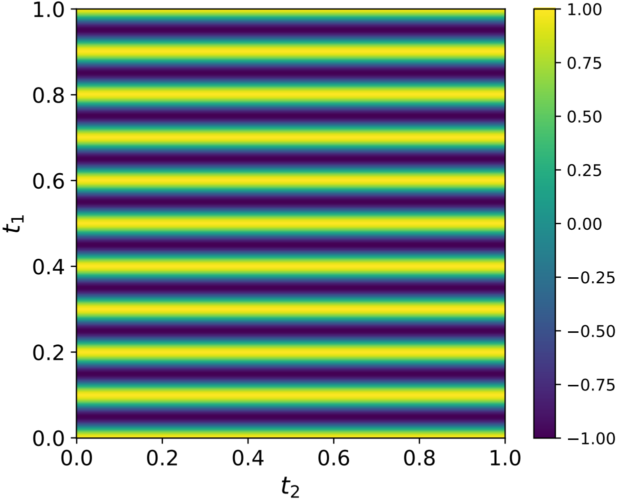
In the direction of f the function is periodic with period $1/\left|\left|f\right|\right|_2$

$$\begin{aligned} a \cos\left(2\pi\left\langle f, t + \frac{1}{\left|\left|f\right|\right|_2}\frac{f}{\left|\left|f\right|\right|_2}\right\rangle + \theta\right) &= a \cos\left(2\pi\left\langle f, t\right\rangle + 2\pi\frac{\langle f, f\rangle}{\left|\left|f\right|\right|_2^2} + \theta\right) \\ &= a \cos\left(2\pi\langle f, t\rangle + 2\pi + \theta\right) \\ &= a \cos\left(2\pi\langle f, t\rangle + \theta\right) \end{aligned}$$

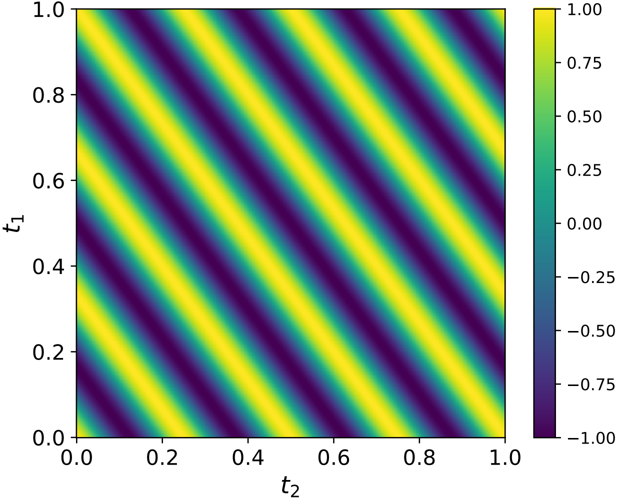
$$\cos\left(2\pi\left\langle\begin{bmatrix}0\\5\end{bmatrix},\begin{bmatrix}t_1\\t_2\end{bmatrix}\right\rangle\right)$$



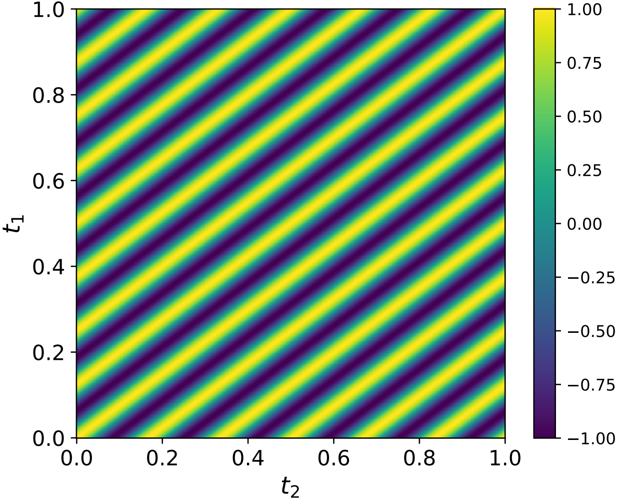
$$\cos\left(2\pi\left\langle \begin{bmatrix}10\\0\end{bmatrix}, \begin{bmatrix}t_1\\t_2\end{bmatrix}\right\rangle\right)$$



$$\cos\left(2\pi\left\langle\begin{bmatrix}3\\4\end{bmatrix},\begin{bmatrix}t_1\\t_2\end{bmatrix}\right\rangle\right)$$



$$\cos\left(2\pi\left\langle\begin{bmatrix}8\\-6\end{bmatrix},\begin{bmatrix}t_1\\t_2\end{bmatrix}\right\rangle\right)$$



Multidimensional complex sinusoids

Complex sinusoid with frequency $f \in \mathbb{R}^d$:

$$\exp(i2\pi\langle f, t \rangle) := \cos(2\pi\langle f, t \rangle) + i \sin(2\pi\langle f, t \rangle)$$

We can express any phase using positive / negative frequencies as in 1D

$$\begin{aligned}
\cos\left(i2\pi\langle f, t \rangle + \theta\right) &= \frac{\exp(i2\pi\langle f, t \rangle + i\theta) + \exp(-i2\pi\langle f, t \rangle - i\theta)}{2} \\
&= \frac{\exp(i\theta)}{2} \exp(i2\pi\langle f, t \rangle) + \frac{\exp(-i\theta)}{2} \exp(-i2\pi\langle f, t \rangle)
\end{aligned}$$

Multidimensional complex sinusoids

Can be expressed as product of 1D complex sinusoids

$$\exp(i2\pi\langle f, t\rangle) := \exp\left(i2\pi\sum_{j=1}^{d} f[j]t[j]\right)$$

$$= \prod_{j=1}^{d} \exp(i2\pi f[j]t[j])$$

2D complex sinusoids

From now on $d = 2$:

$$f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

$$t = \begin{bmatrix} t_1 \\ t_2 \end{bmatrix}$$

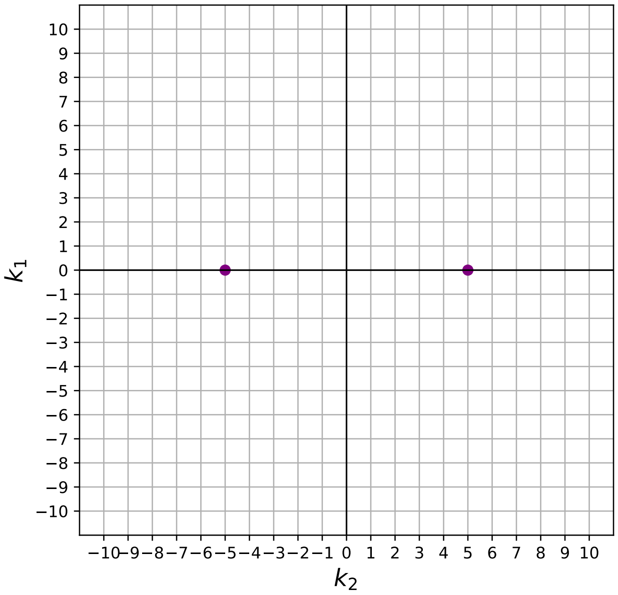
$$\begin{aligned} \exp(i2\pi\langle f, t \rangle) &:= \exp\left(i2\pi(f_1 t_1 + f_2 t_2)\right) \\ &= \exp(i2\pi f_1 t_1) \exp(i2\pi f_2 t_2) \end{aligned}$$

2D complex sinusoids

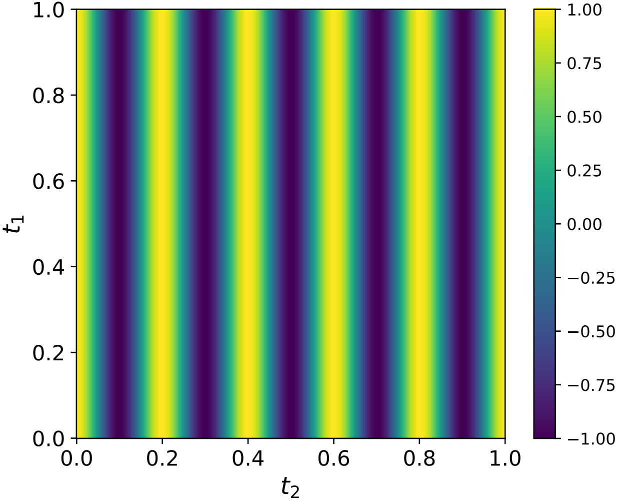
Family of complex sinusoids on $[0, T) \times [0, T)$

$$\phi^{2D}_{k_1,k_2}(t_1, t_2) := \exp\left(\frac{i2\pi k_1 t_1}{T}\right)\exp\left(\frac{i2\pi k_2 t_2}{T}\right), \qquad k_1, k_2 \in \mathbb{Z}$$

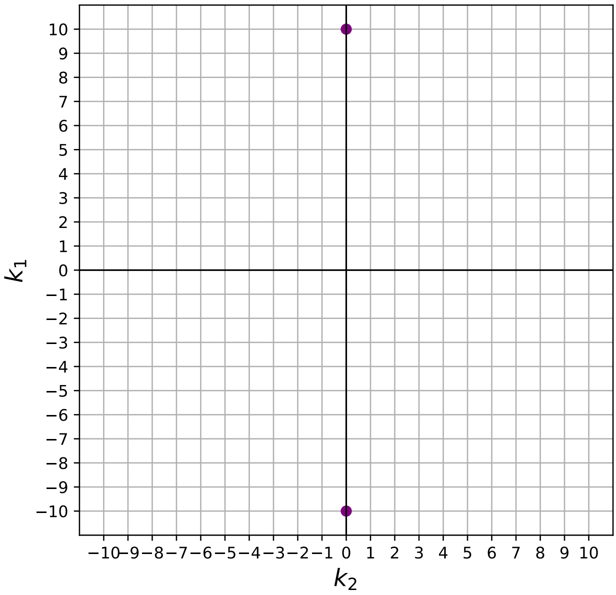
$$\frac{\phi^{2D}_{0,5}+\phi^{2D}_{0,-5}}{2}$$



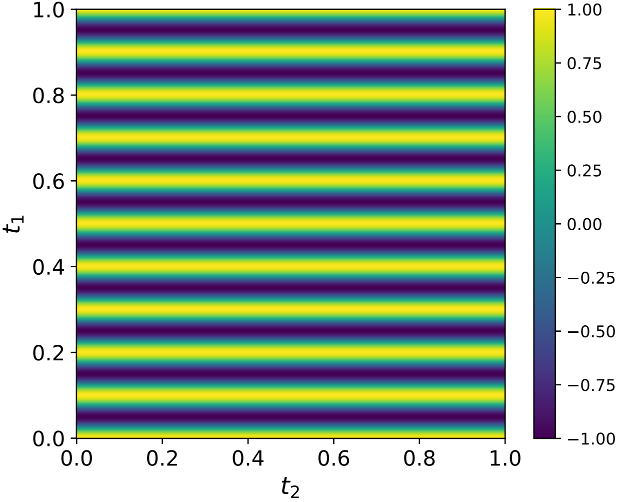
$$\frac{\phi^{2D}_{0,5}+\phi^{2D}_{0,-5}}{2}$$



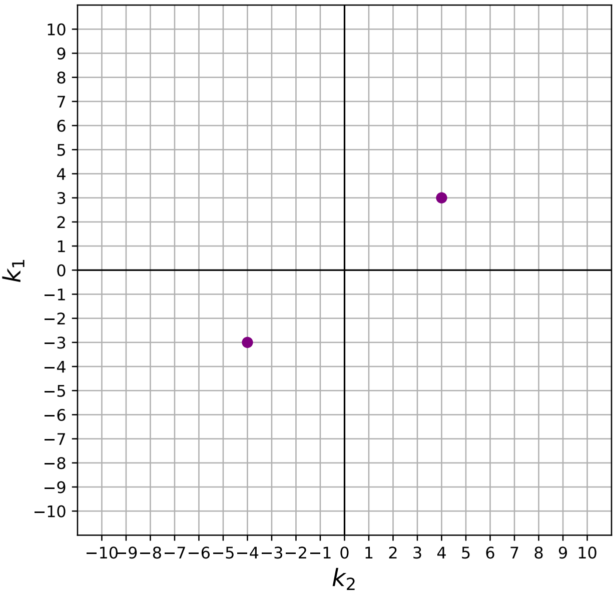
$$\frac{\phi^{2D}_{10,0}+\phi^{2D}_{-10,0}}{2}$$



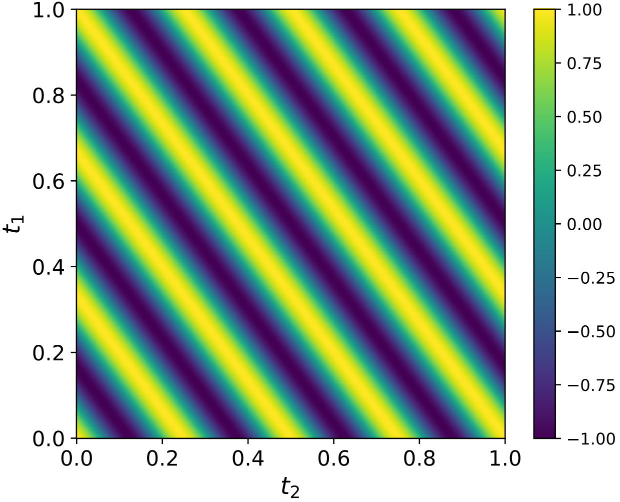
$$\frac{\phi^{\mathrm{2D}}_{10,0}+\phi^{\mathrm{2D}}_{-10,0}}{2}$$



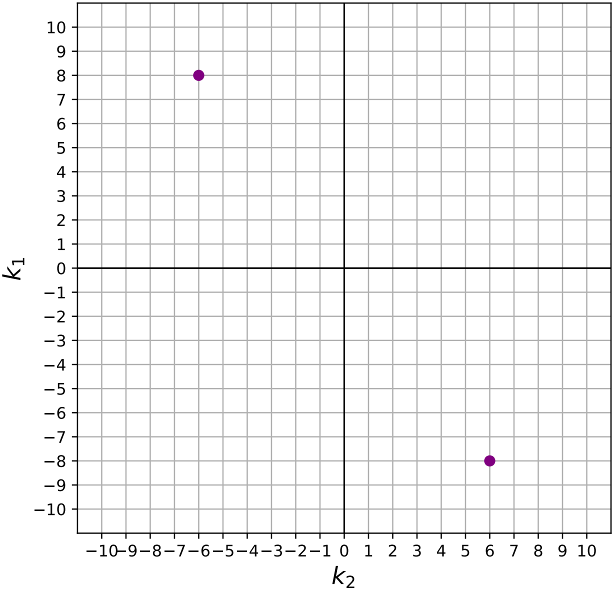
$$\frac{\phi^{2D}_{3,4} + \phi^{2D}_{-3,-4}}{2}$$



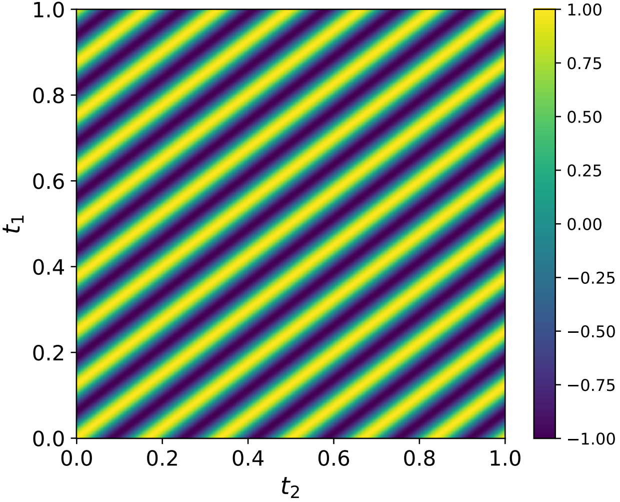
$$\frac{\phi^{2D}_{3,4}+\phi^{2D}_{-3,-4}}{2}$$



$$\frac{\phi^{2D}_{8,-6}+\phi^{2D}_{-8,6}}{2}$$



$$\frac{\phi^{2D}_{8,-6}+\phi^{2D}_{-8,6}}{2}$$



Orthogonality of multidimensional complex sinusoids

The family of complex sinusoids with integer frequencies

$$\phi^{2D}_{k_1,k_2}(t_1, t_2) := \exp\left(\frac{i2\pi k_1 t_1}{T}\right)\exp\left(\frac{i2\pi k_2 t_2}{T}\right), \qquad k_1, k_2 \in \mathbb{Z},$$

is an orthogonal set of functions on the interval $[0, T] \times [0, T]$

Proof

We have

$$\phi^{\mathrm{2D}}_{k_1,k_2}(t_1,t_2) = \phi_{k_1}(t_1)\,\phi_{k_2}(t_2),$$

so that

$$\begin{aligned}\left\langle \phi^{\mathrm{2D}}_{k_1,k_2}, \phi^{\mathrm{2D}}_{j_1,j_2} \right\rangle &= \int_{t_1=0}^{T}\int_{t_2=0}^{T} \phi_{k_1}(t_1)\,\phi_{k_2}(t_2)\,\overline{\phi_{j_1}(t_1)\,\phi_{j_2}(t_2)}\,\mathrm{d}t_1\,\mathrm{d}t_2 \\ &= \langle \phi_{k_1}, \phi_{j_1}\rangle \langle \phi_{k_2}, \phi_{j_2}\rangle \\ &= 0\end{aligned}$$

as long as $j_1 \neq k_1$ or $j_2 \neq k_2$

If $j_1 = k_1$ and $j_2 = k_2$

$$\begin{aligned}\left\langle \phi^{\mathrm{2D}}_{k_1,k_2}, \phi^{\mathrm{2D}}_{k_1,k_2} \right\rangle &= \langle \phi_{k_1}, \phi_{k_1}\rangle \langle \phi_{k_2}, \phi_{k_2}\rangle \\ &= T^2 \qquad \text{so } \left|\left|\phi^{\mathrm{2D}}_{k_1,k_2}\right|\right| = T\end{aligned}$$

2D Fourier series

Fourier series coefficients of a function $x \in \mathcal{L}_2\left[0, T\right]$

$$\begin{aligned}\hat{x}[k_1, k_2] &:= \left\langle x, \phi^{2\text{D}}_{k_1,k_2} \right\rangle \\ &= \int_{t_1=0}^{T} \int_{t_2=0}^{T} x(t_1, t_2) \exp\left(-\frac{i2\pi k_1 t_1}{T}\right) \exp\left(-\frac{i2\pi k_2 t_2}{T}\right) \, \mathrm{d}t_1 \, \mathrm{d}t_2\end{aligned}$$

The Fourier series of order k_{c1}, k_{c2} is defined as

$$\mathcal{F}_{k_{c1},k_{c2}}\left\{x\right\} := \frac{1}{T^2} \sum_{k_1=-k_{c1}}^{k_{c1}} \sum_{k_2=-k_{c2}}^{k_{c2}} \hat{x}[k_1, k_2] \phi^{2\text{D}}_{k_1,k_2}.$$

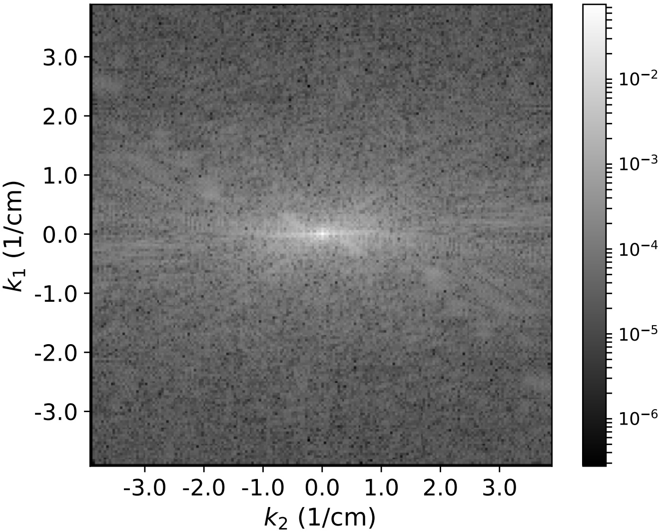
Magnetic resonance imaging

Non-invasive medical-imaging technique

Measures response of atomic nuclei in biological tissues to high-frequency radio waves when placed in a strong magnetic field

Radio waves adjusted so that each measurement equals 2D Fourier coefficients of proton density of hydrogen atoms in a region of interest

Data



3.0
2.0
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k_1 (1/cm)
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k_2 (1/cm)
10^{-2}
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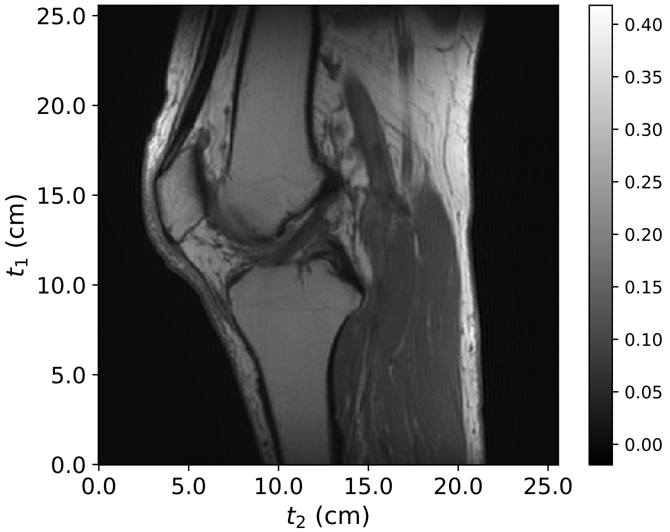
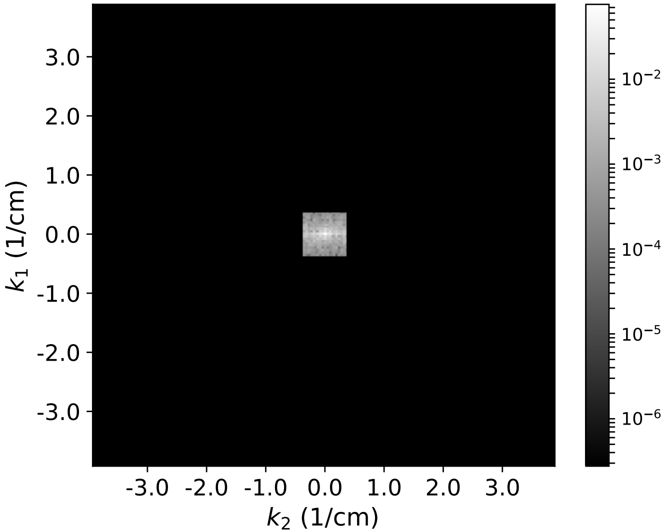
Recovered image



Data



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k_1 (1/cm)
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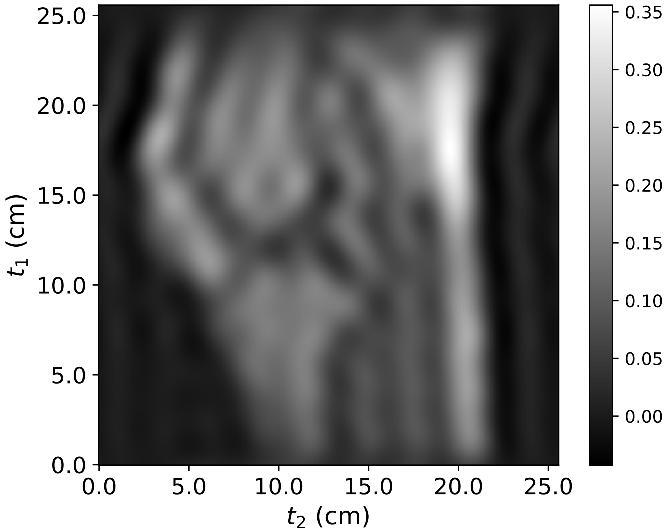

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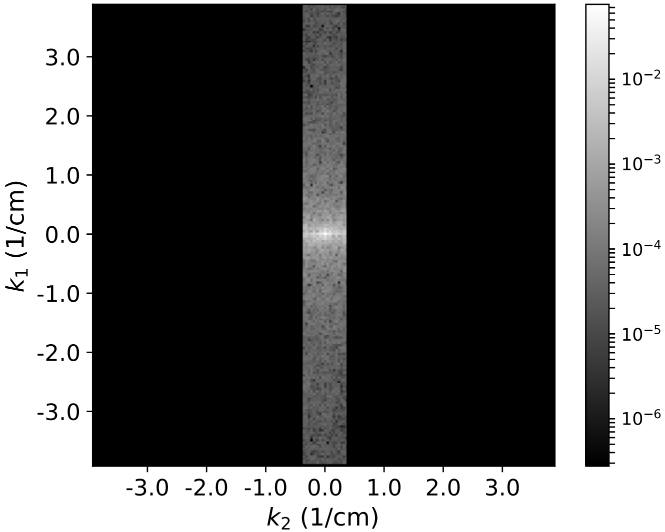

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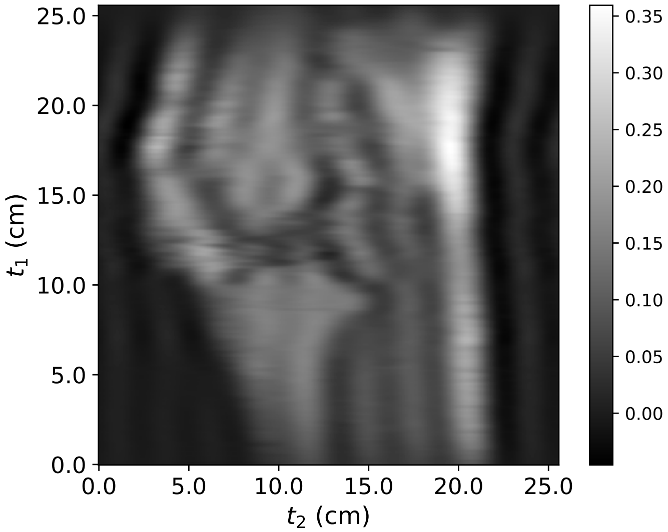
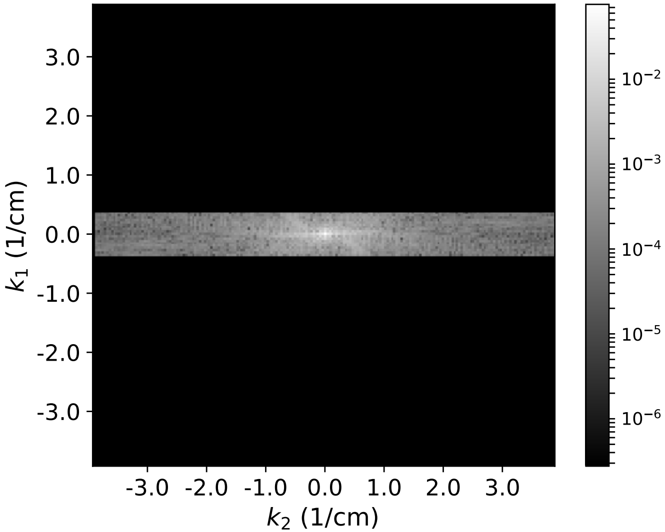


Data



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k_1 (1/cm)
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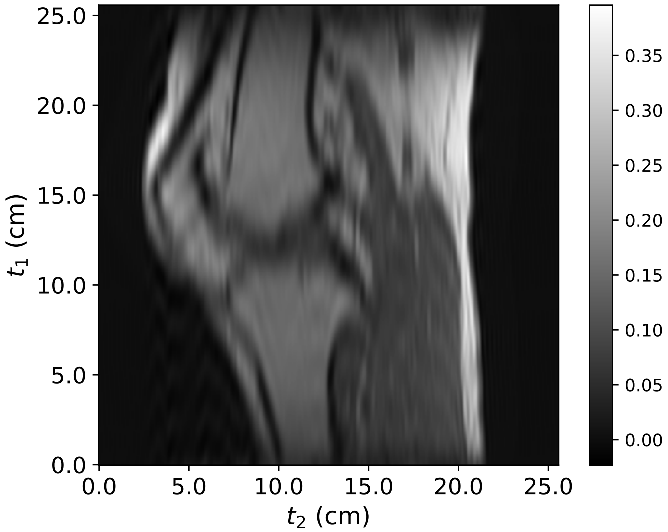

Bandlimited signal

A signal defined on the 2D rectangle $[0, T] \times [0, T]$ is bandlimited with a cut-off frequency k_c if

$$x(t_1, t_2) = \frac{1}{T^2} \sum_{k_1=-k_c}^{k_c} \sum_{k_2=-k_c}^{k_c} \hat{x}[k_1, k_2] \exp\left(\frac{i2\pi k_1 t_1}{T}\right) \exp\left(\frac{i2\pi k_2 t_2}{T}\right)$$

Equispaced grid

$$X_{[N]} := \begin{bmatrix} x\left(\frac{0}{N}, \frac{0}{N}\right) & x\left(\frac{0}{N}, \frac{T}{N}\right) & \cdots & x\left(\frac{0}{N}, T - \frac{T}{N}\right) \\ x\left(\frac{T}{N}, \frac{0}{N}\right) & x\left(\frac{T}{N}, \frac{T}{N}\right) & \cdots & x\left(\frac{T}{N}, T - \frac{T}{N}\right) \\ \cdots & \cdots & \cdots & \cdots \\ x\left(T - \frac{T}{N}, \frac{0}{N}\right) & x\left(T - \frac{T}{N}, \frac{T}{N}\right) & \cdots & x\left(T - \frac{T}{N}, T - \frac{T}{N}\right) \end{bmatrix}.$$

Nyquist-Shannon-Kotelnikov sampling theorem

Any bandlimited signal $x \in \mathcal{L}_2[0, T)^2$, where $T > 0$, with cut-off frequency k_c can be recovered from N^2 uniformly spaced samples if

$$N \geq 2k_c + 1,$$

where $2k_c + 1$ is known as the *Nyquist rate*

2D discrete signals

We represent 2D signals as matrices belonging to the vector space of $\mathbb{C}^{N\times N}$ matrices endowed with the standard inner product

$$\langle A, B\rangle := \operatorname{tr}\left(A^* B\right), \quad A, B \in \mathbb{C}^{N\times N}$$

Equivalent to dot product between vectorized matrices

Discrete complex sinusoids

The discrete complex sinusoid $\psi^{2D}_{k_1,k_2} \in \mathbb{C}^{N\times N}$ with integer frequencies k_1 and k_2 is defined as

$$\psi^{2D}_{k_1,k_2}[j_1,j_2] := \exp\left(\frac{i2\pi k_1 j_1}{N}\right)\exp\left(\frac{i2\pi k_2 j_2}{N}\right), \qquad 0 \le j_1, j_2 \le N-1,$$

Equivalently

$$\psi^{2D}_{k_1,k_2} = \psi_{k_1}\psi_{k_2}^T$$

The discrete complex exponentials $\frac{1}{N}\psi^{2D}_{k_1,k_2}$, $0 \le k_1, k_2 \le N-1$, form an orthonormal basis of $\mathbb{C}^{N\times N}$

Proof

$$\left\langle \psi^{2D}_{k_1,k_2}, \psi^{2D}_{l_1,l_2} \right\rangle = \mathrm{tr}\left(\left(\psi^{2D}_{l_1,l_2} \right)^* \psi^{2D}_{k_1,k_2} \right)$$
$$= (\psi_{k_1})^* \psi_{l_1} (\psi_{k_2})^* \psi_{l_2}$$

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$$N \geq 2k_c + 1,$$

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$$\begin{aligned}\hat{x}[k_1, k_2] &= \frac{T}{N^2}\left\langle \psi^{2\mathrm{D}}_{k_1,k_2}, X_{[N]} \right\rangle \\ &= \frac{T}{N^2}\psi^*_{k_1} X_{[N]} \psi_{k_2}.\end{aligned}$$

Proof

For any m_1, m_2, we have $\phi^{2D}_{m_1,m_2}\left[\frac{j_1 T}{N}, \frac{j_2 T}{N}\right] = \psi^{2D}_{m_1,m_2}[j_1, j_2]$, so

$$\begin{aligned}
\frac{T}{N^2}\left\langle \psi^{2D}_{k_1,k_2}, X_{[N]} \right\rangle &= \frac{T}{N^2}\left\langle \psi^{2D}_{k_1,k_2}, \frac{1}{T}\sum_{m_1=-k_c}^{k_c}\sum_{m_2=-k_c}^{k_c} \hat{x}[m_1, m_2]\psi^{2D}_{m_1,m_2} \right\rangle \\
&= \sum_{m_1=-k_c}^{k_c}\sum_{m_2=-k_c}^{k_c} \hat{x}[m_1, m_2]\left\langle \frac{1}{N}\psi^{2D}_{k_1,k_2}, \frac{1}{N}\psi^{2D}_{m_1,m_2} \right\rangle \\
&= \hat{x}[k_1, k_2]
\end{aligned}$$

2D discrete Fourier transform

The discrete Fourier transform (DFT) of a 2D array $X \in \mathbb{C}^{N \times N}$ is

$$\widehat{X}[k_1, k_2] := \left\langle X, \psi^{\text{2D}}_{k_1,k_2} \right\rangle, \qquad 0 \leq k_1, k_2 \leq N-1,$$

or equivalently

$$\widehat{X} := F_{[N]} X \, F_{[N]},$$

where $F_{[N]}$ is the 1D DFT matrix

It can be computed efficiently with the FFT (complexity $N^2 \log N$)

Inverse 2D discrete Fourier transform

The inverse DFT of a 2D array $\widehat{Y} \in \mathbb{C}^{N \times N}$ equals

$$Y = \frac{1}{N^2} F_{[N]}^* \widehat{Y} F_{[N]}^*$$

It inverts the 2D DFT

What have we learned

Properties of multidimensional sinusoids

Definition of multidimensional Fourier series

Definition of multidimensional DFT

Sampling theorem in 2D