Fourier series and sampling

(blended lecture)

DS-GA 1013 / MATH-GA 2824 Mathematical Tools for Data Science

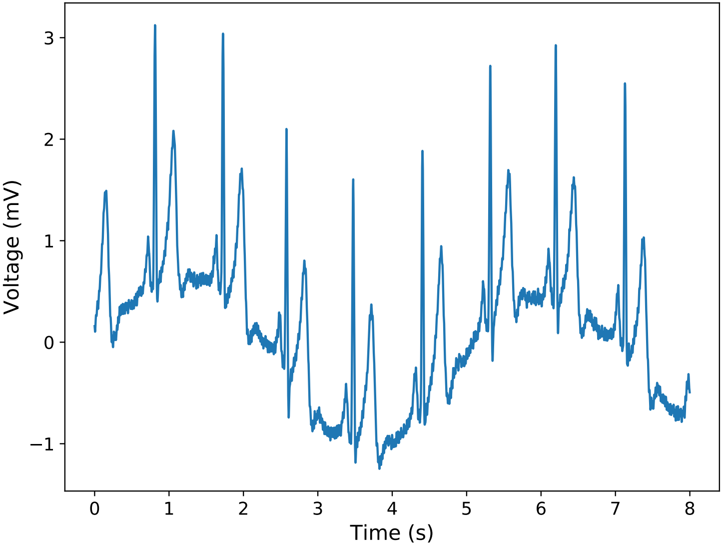
Carlos Fernandez-Granda

Fourier series

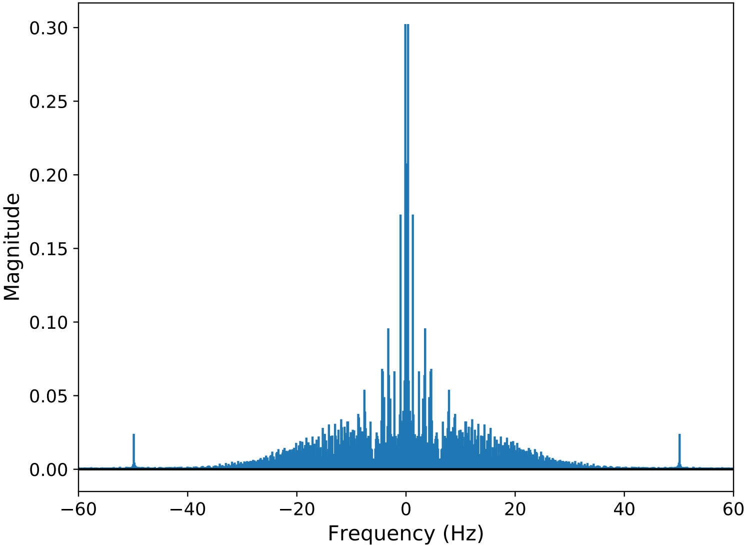
Filtering

Sampling

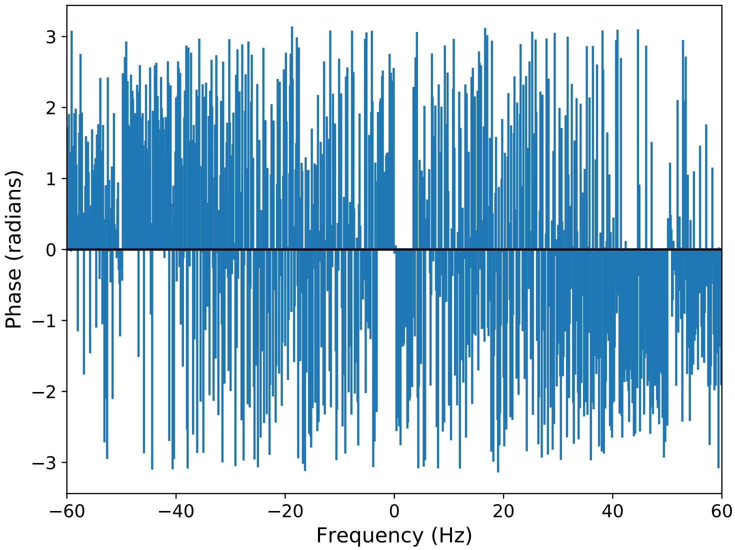
Electrocardiogram



Electrocardiogram: Fourier coefficients (magnitude)



Electrocardiogram: Fourier coefficients (phase)



Real valued signals

Why is the magnitude even and the phase odd? Is this a coincidence?

Let $\hat{x}[k] = \alpha_k \exp(i\xi_k)$, do we always have $\hat{x}[-k] = \alpha_k \exp(-i\xi_k)$?

Real valued signals

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Let $\hat{x}[k] = \alpha_k \exp(i\xi_k)$, do we always have $\hat{x}[-k] = \alpha_k \exp(-i\xi_k)$?

$$\hat{x}[-k] := \int_0^T x(t) \exp\left(\frac{i2\pi kt}{T}\right) \mathrm{d}t$$

$$= \overline{\int_0^T x(t) \exp\left(\frac{-i2\pi kt}{T}\right) \mathrm{d}t} \quad \text{because } x \text{ is real valued}$$

$$= \overline{\hat{x}[k]}$$

$$= \alpha_k \exp(-i\xi_k)$$

Real sinusoids

Can we express the Fourier series of a real signal in terms of real sinusoids?

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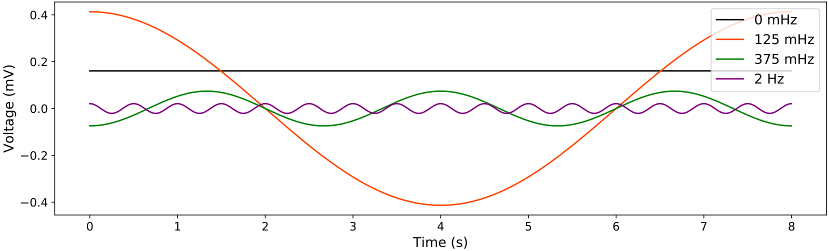
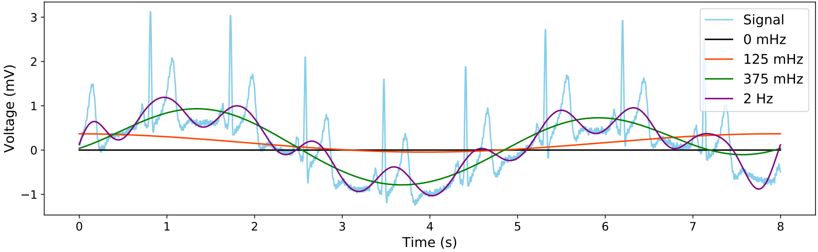
$$\begin{aligned}
&\hat{x}[k]\phi_k + \hat{x}[-k]\phi_{-k} \\
&= \alpha_k \exp\left(i\xi_k\right) \exp\left(\frac{i2\pi kt}{T}\right) + \alpha_k \exp\left(-i\xi_k\right) \exp\left(-\frac{i2\pi kt}{T}\right) \\
&= \alpha_k \left(\exp\left(\frac{i2\pi kt}{T} + i\xi_k\right) + \exp\left(-\frac{i2\pi kt}{T} - i\xi_k\right)\right) \\
&= 2\alpha_k \cos\left(\frac{2\pi kt}{T} + \xi_k\right)
\end{aligned}$$

$$\mathcal{F}_{k_c}\left\{x\right\} := \frac{1}{T}\sum_{k=-k_c}^{k_c} \hat{x}[k]\phi_k = \alpha_0 + \frac{1}{T}\sum_{k=0}^{k_c} 2\alpha_k \cos\left(\frac{2\pi kt}{T} + \xi_k\right)$$

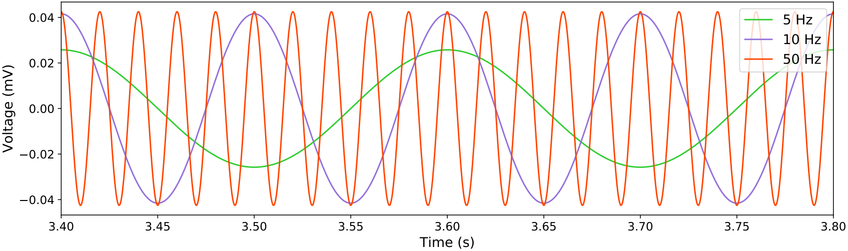
Fourier components



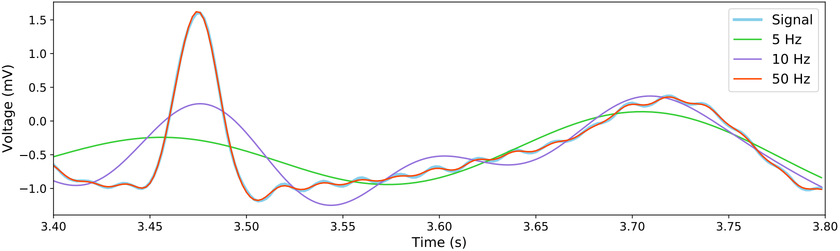
Fourier series



Fourier components



Fourier series

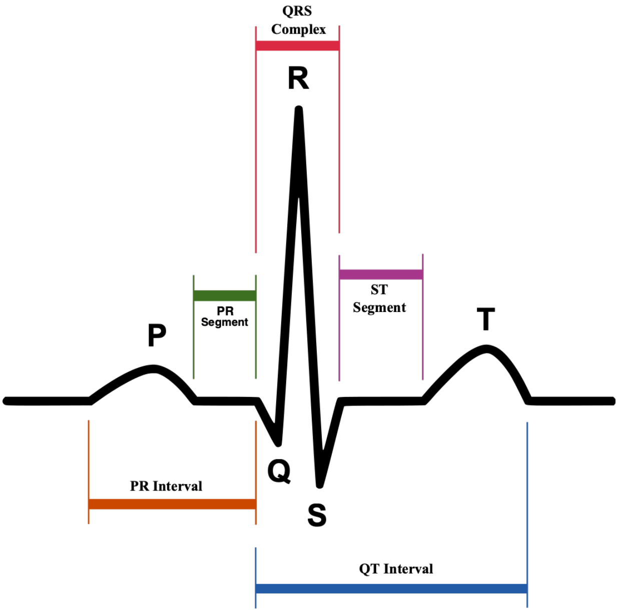


Fourier series

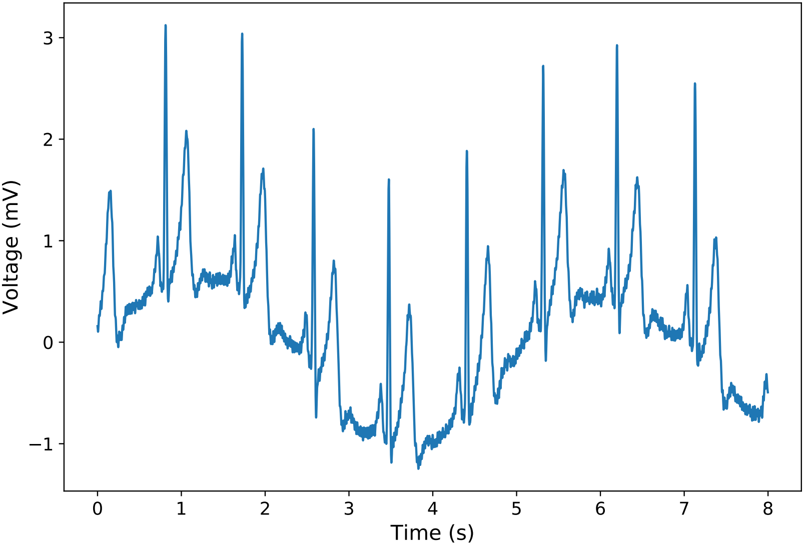
Filtering

Sampling

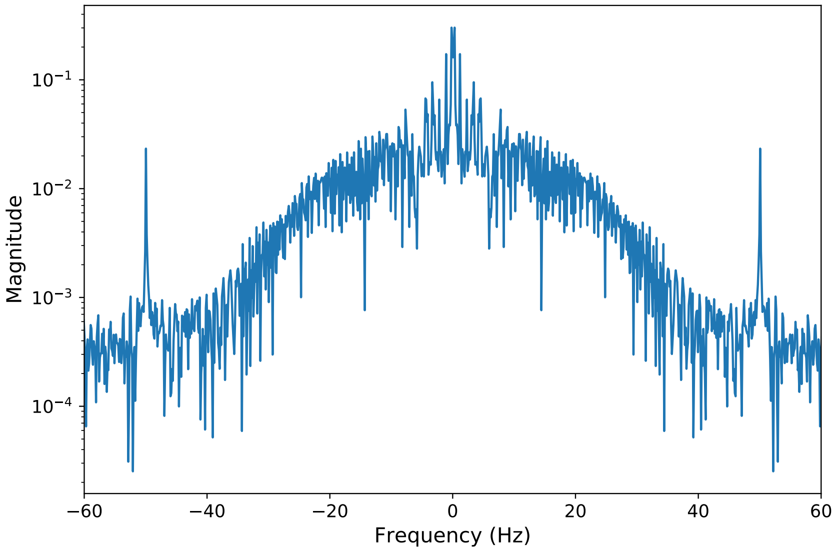
Electrocardiogram features



Problem: Baseline wandering



Electrocardiogram: Fourier coefficients (magnitude)



Filtering

Idea: Process signal by removing (or attenuating) frequency components

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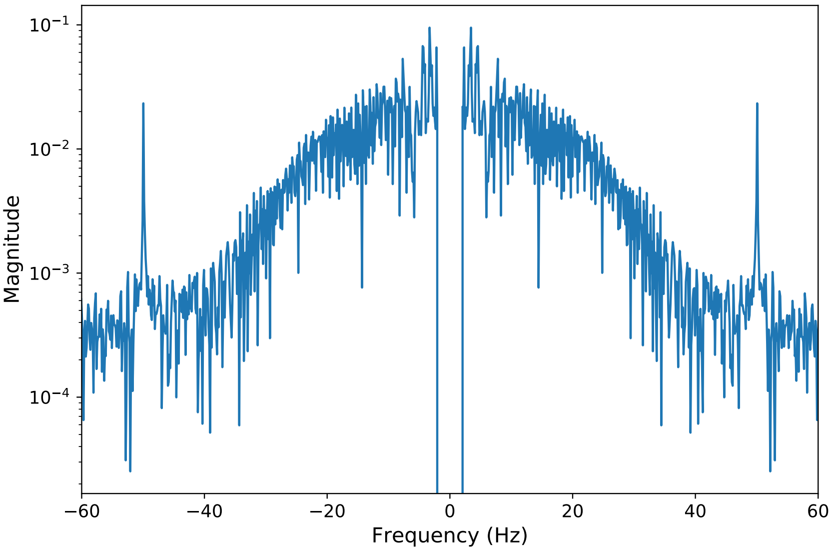
High-pass filtering to correct baseline wandering

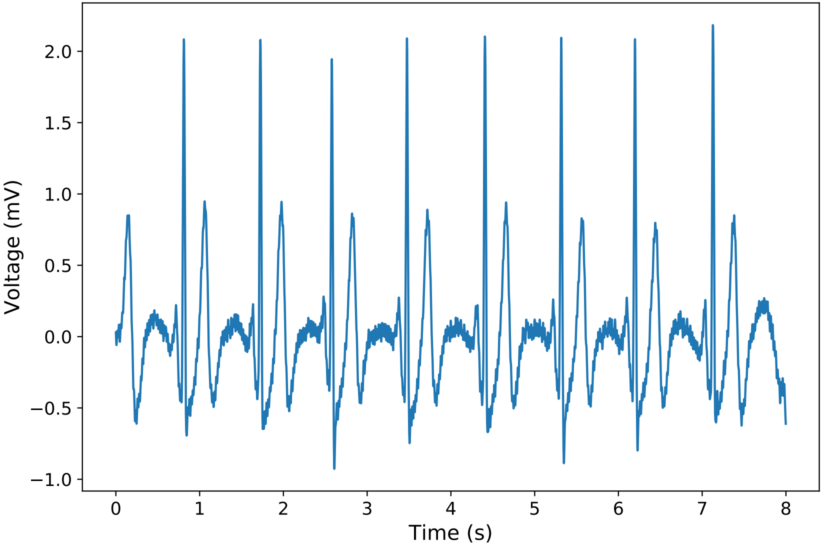
$$x \approx \frac{1}{T} \sum_{k=-k_{\text{max}}}^{-k_{\text{max}}} \hat{x}[k] \phi_k$$

$$x_{\text{high-pass}} := \frac{1}{T} \sum_{k=-k_{\text{max}}}^{-k_{\text{thresh}}} \hat{x}[k] \phi_k + \frac{1}{T} \sum_{k=k_{\text{thresh}}}^{k_{\text{max}}} \hat{x}[k] \phi_k$$

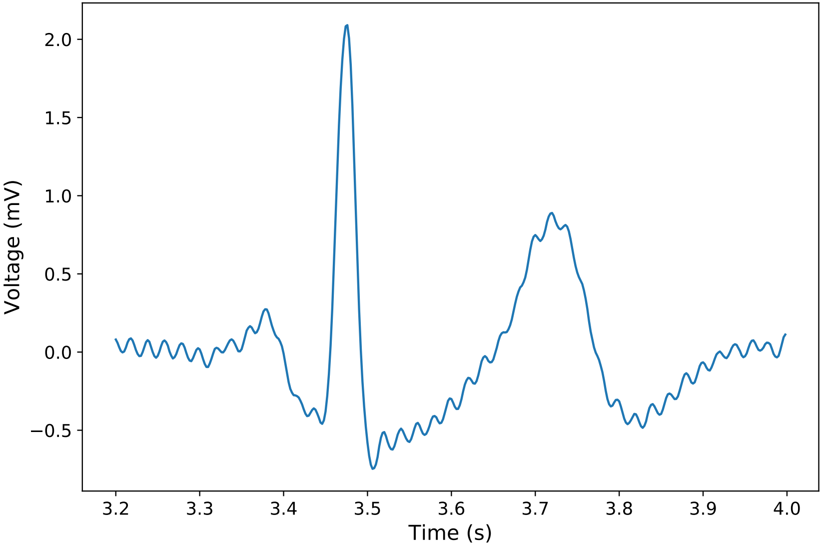
Electrocardiogram after high-pass filtering



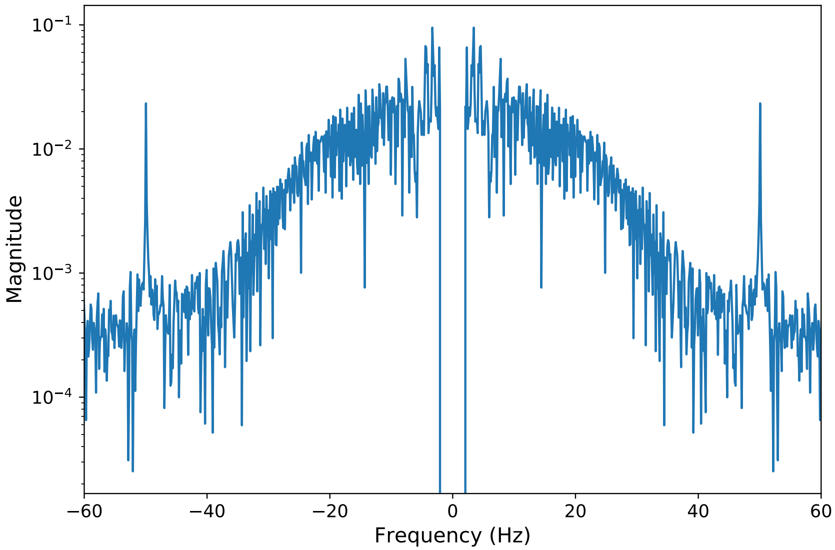
Electrocardiogram after high-pass filtering



Problem: Electric-grid interference



Fourier coefficients (magnitude)



Filtering

Idea: Can we remove the interference by filtering?

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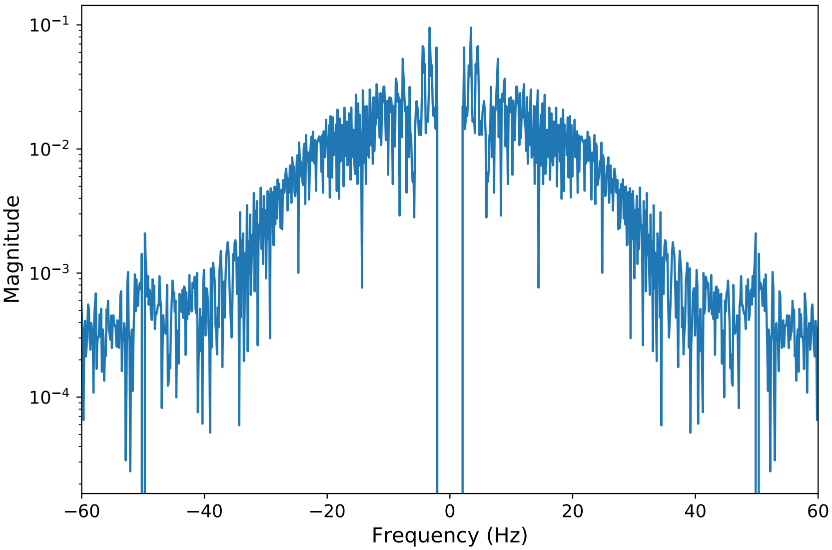
Band-stop filtering

$$x_{\text{filtered}} := \frac{1}{T} \sum_{k=-k_{\text{max}}}^{-k_{\text{band-end}}} \hat{x}[k]\phi_k + \frac{1}{T} \sum_{k=-k_{\text{band-ini}}}^{-k_{\text{thresh}}} \hat{x}[k]\phi_k$$

$$+ \frac{1}{T} \sum_{k=k_{\text{thresh}}}^{k_{\text{band-ini}}} \hat{x}[k]\phi_k + \frac{1}{T} \sum_{k=k_{\text{band-end}}}^{k_{\text{max}}} \hat{x}[k]\phi_k$$

Filtered electrocardiogram



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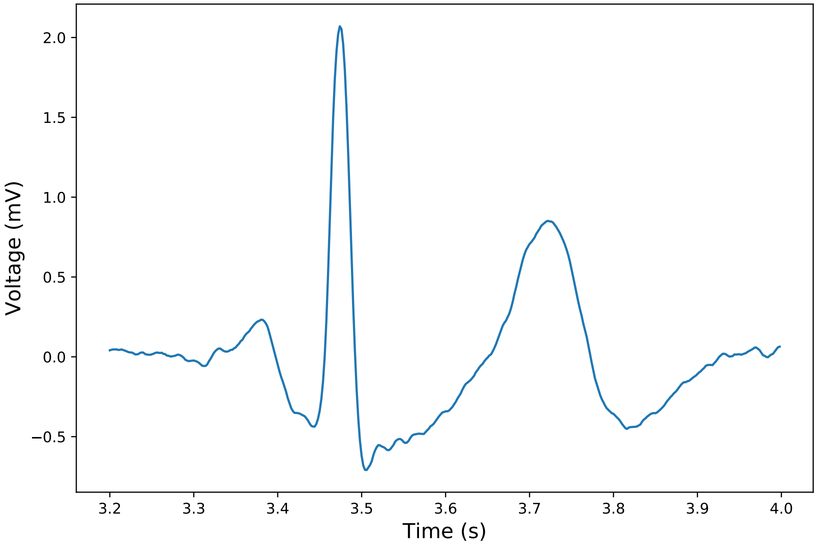
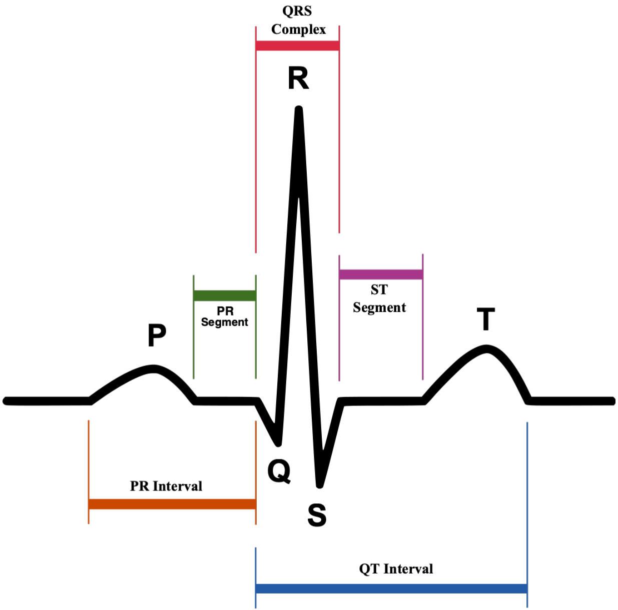


Electrocardiogram features



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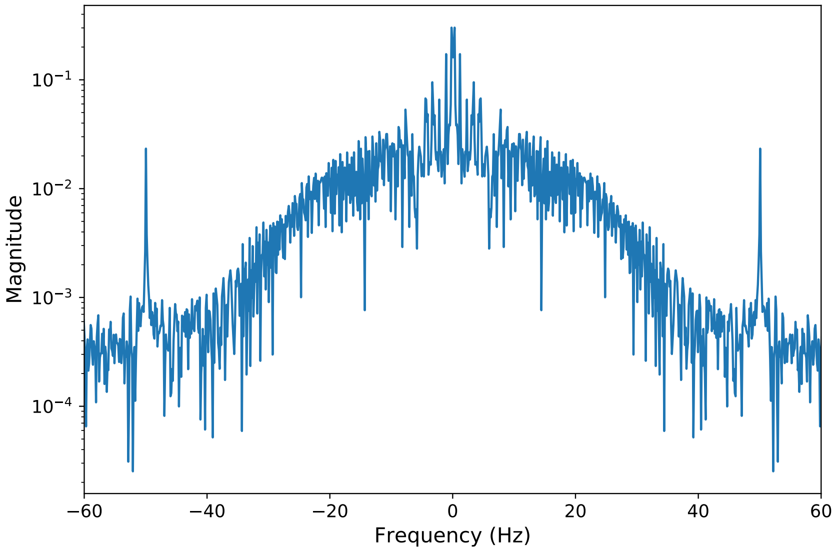
Watch videos

Electrocardiogram: Fourier coefficients (magnitude)

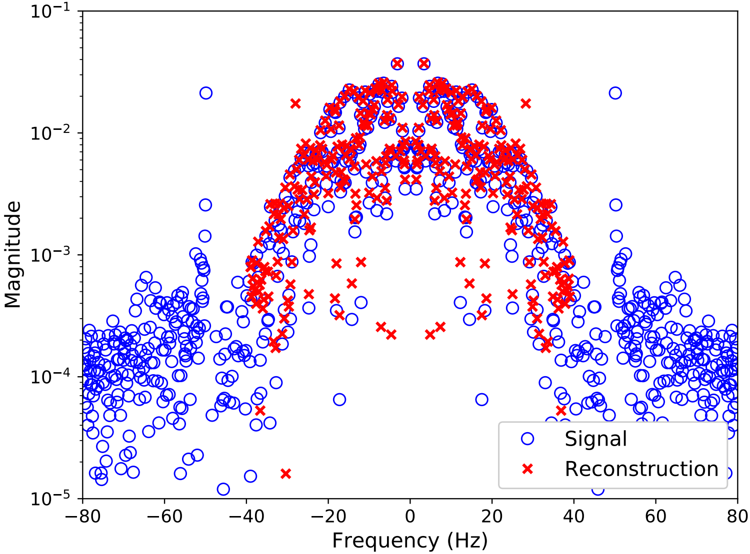


Sampling an electrocardiogram

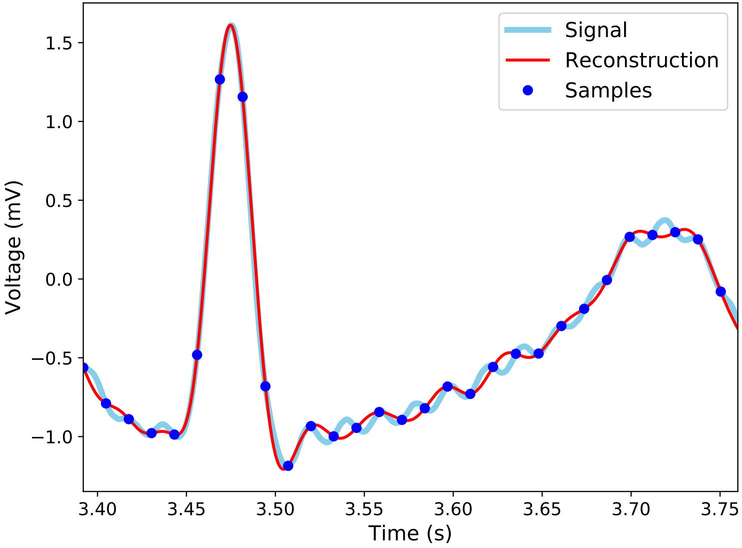
We are only interested in frequencies up to around 40 Hz, so we can choose $k_c = 312$ and $N = 625$

Will this work?

Recovered Fourier coefficients ($N = 625$)



Recovered signal ($N = 625$)



What is going on?

Why do we see a phantom component at ± 225 (28.1 Hz)?

Recall that $N = 625$ and powerline interference is at ± 400 (50 Hz)

$$\hat{x}^{\text{rec}}[225] =$$

What is going on?

Why do we see a phantom component at ± 225 (28.1 Hz)?

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$$\begin{aligned}
\hat{x}^{\text{rec}}[225] &= \frac{T}{N}\left\langle \psi_k, x_{[N]} \right\rangle \\
&= \frac{T}{N}\left\langle \frac{1}{T}\sum_{m=-k_{\max}}^{k_{\max}} \hat{x}[m]\psi_m, \psi_k \right\rangle \\
&= \frac{1}{N}\sum_{m=-k_{\max}}^{k_{\max}} \hat{x}[m]\left\langle \psi_m, \psi_k \right\rangle \\
&= \sum_{\{(m-k) \bmod N = 0\}} \hat{x}[m] \\
&\approx \hat{x}[225] + \hat{x}[-400]
\end{aligned}$$