



The frequency domain (2nd blended lecture)

DS-GA 1013 / MATH-GA 2824 Mathematical Tools for Data Science

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The fast Fourier transform

2D Fourier series

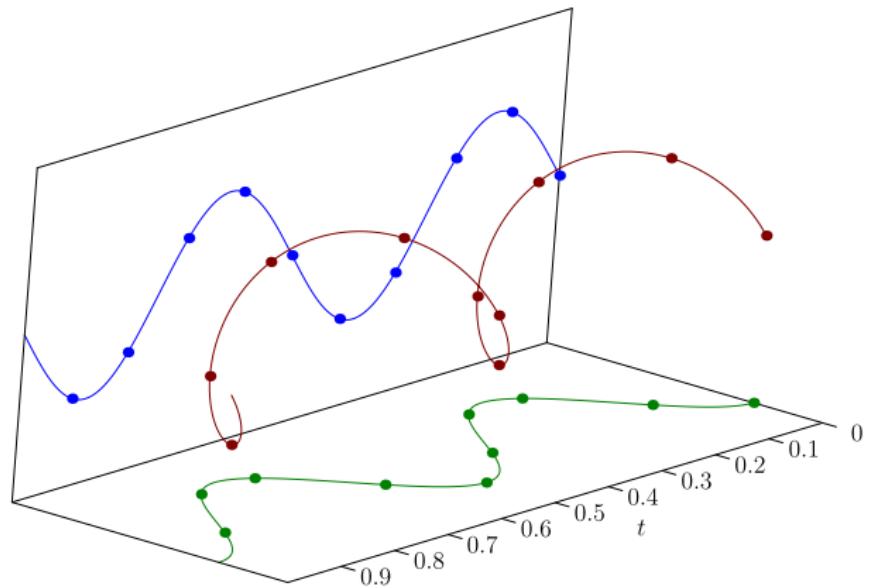
Discrete complex sinusoids

The discrete complex sinusoid $\psi_k \in \mathbb{C}^N$ with frequency k is

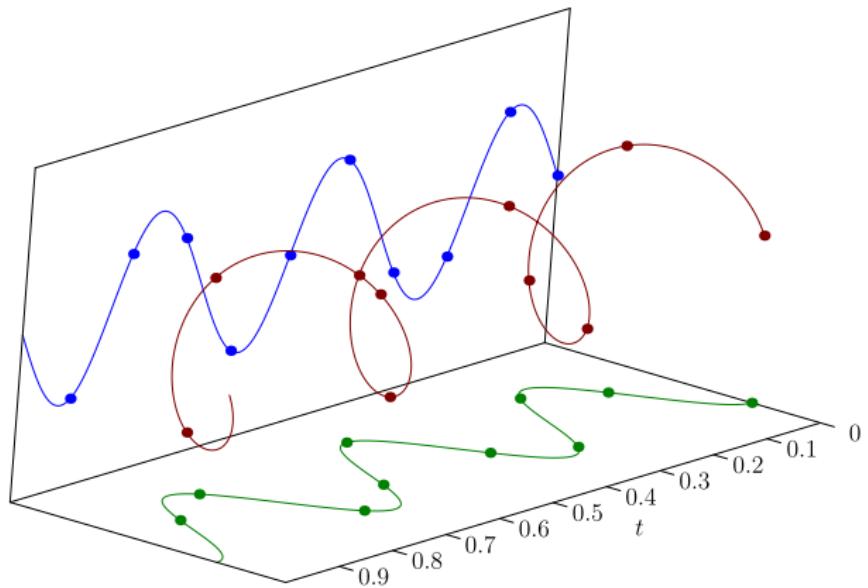
$$\psi_k[j] := \exp\left(\frac{i2\pi kj}{N}\right), \quad 0 \leq j, k \leq N-1$$

Discrete complex sinusoids scaled by $1/\sqrt{N}$: orthonormal basis of \mathbb{C}^N

ψ_2 ($N=10$)



ψ_3 ($N=10$)



Discrete Fourier transform

The discrete Fourier transform (DFT) of $x \in \mathbb{C}^N$ is

$$\hat{x} := \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \exp\left(-\frac{i2\pi}{N}\right) & \exp\left(-\frac{i2\pi 2}{N}\right) & \cdots & \exp\left(-\frac{i2\pi(N-1)}{N}\right) \\ 1 & \exp\left(-\frac{i2\pi 2}{N}\right) & \exp\left(-\frac{i2\pi 4}{N}\right) & \cdots & \exp\left(-\frac{i2\pi 2(N-1)}{N}\right) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & \exp\left(-\frac{i2\pi(N-1)}{N}\right) & \exp\left(-\frac{i2\pi 2(N-1)}{N}\right) & \cdots & \exp\left(-\frac{i2\pi(N-1)^2}{N}\right) \end{bmatrix} x$$
$$= F_{[N]}x$$

$$\hat{x}[k] = \langle x, \psi_k \rangle, \quad 0 \leq k \leq N - 1$$

Complexity of computing the DFT

Complexity of multiplying $N \times N$ matrix with N -dimensional vector is N^2

Very slow!

We can exploit the structure of the matrix to do much better

Fast Fourier transform

The most important numerical algorithm of our lifetime (G. Strang)

Main insight:

$N \times N$ DFT matrix can be expressed in terms of $\frac{N}{2} \times \frac{N}{2}$ DFT matrices

Separation in even/odd columns and top/bottom rows

$$\begin{bmatrix} \hat{x}[0] \\ \hat{x}[1] \\ \hat{x}[2] \\ \hat{x}[3] \\ \hat{x}[4] \\ \hat{x}[5] \\ \hat{x}[6] \\ \hat{x}[7] \end{bmatrix} = \begin{bmatrix} \text{dark blue} & \text{green} & \text{dark blue} & \text{green} & \text{dark blue} & \text{green} & \text{dark blue} & \text{green} \\ \text{light blue} & \text{light green} & \text{light blue} & \text{light green} & \text{light blue} & \text{light green} & \text{light blue} & \text{light green} \end{bmatrix} \begin{bmatrix} \vec{x}[0] \\ \vec{x}[1] \\ \vec{x}[2] \\ \vec{x}[3] \\ \vec{x}[4] \\ \vec{x}[5] \\ \vec{x}[6] \\ \vec{x}[7] \end{bmatrix}$$

Example $N = 8$

$$\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_7 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & e^{-\frac{i2\pi}{8}} & e^{-\frac{i2\pi \cdot 2}{8}} & e^{-\frac{i2\pi \cdot 3}{8}} & e^{-\frac{i2\pi \cdot 4}{8}} & e^{-\frac{i2\pi \cdot 5}{8}} & e^{-\frac{i2\pi \cdot 6}{8}} & e^{-\frac{i2\pi \cdot 7}{8}} \\ 1 & e^{-\frac{i2\pi \cdot 2}{8}} & e^{-\frac{i2\pi \cdot 2 \cdot 2}{8}} & e^{-\frac{i2\pi \cdot 2 \cdot 3}{8}} & e^{-\frac{i2\pi \cdot 2 \cdot 4}{8}} & e^{-\frac{i2\pi \cdot 2 \cdot 5}{8}} & e^{-\frac{i2\pi \cdot 2 \cdot 6}{8}} & e^{-\frac{i2\pi \cdot 2 \cdot 7}{8}} \\ 1 & e^{-\frac{i2\pi \cdot 3}{8}} & e^{-\frac{i2\pi \cdot 3 \cdot 2}{8}} & e^{-\frac{i2\pi \cdot 3 \cdot 3}{8}} & e^{-\frac{i2\pi \cdot 3 \cdot 4}{8}} & e^{-\frac{i2\pi \cdot 3 \cdot 5}{8}} & e^{-\frac{i2\pi \cdot 3 \cdot 6}{8}} & e^{-\frac{i2\pi \cdot 3 \cdot 7}{8}} \\ 1 & e^{-\frac{i2\pi \cdot 4}{8}} & e^{-\frac{i2\pi \cdot 4 \cdot 2}{8}} & e^{-\frac{i2\pi \cdot 4 \cdot 3}{8}} & e^{-\frac{i2\pi \cdot 4 \cdot 4}{8}} & e^{-\frac{i2\pi \cdot 4 \cdot 5}{8}} & e^{-\frac{i2\pi \cdot 4 \cdot 6}{8}} & e^{-\frac{i2\pi \cdot 4 \cdot 7}{8}} \\ 1 & e^{-\frac{i2\pi \cdot 5}{8}} & e^{-\frac{i2\pi \cdot 5 \cdot 2}{8}} & e^{-\frac{i2\pi \cdot 5 \cdot 3}{8}} & e^{-\frac{i2\pi \cdot 5 \cdot 4}{8}} & e^{-\frac{i2\pi \cdot 5 \cdot 5}{8}} & e^{-\frac{i2\pi \cdot 5 \cdot 6}{8}} & e^{-\frac{i2\pi \cdot 5 \cdot 7}{8}} \\ 1 & e^{-\frac{i2\pi \cdot 6}{8}} & e^{-\frac{i2\pi \cdot 6 \cdot 2}{8}} & e^{-\frac{i2\pi \cdot 6 \cdot 3}{8}} & e^{-\frac{i2\pi \cdot 6 \cdot 4}{8}} & e^{-\frac{i2\pi \cdot 6 \cdot 5}{8}} & e^{-\frac{i2\pi \cdot 6 \cdot 6}{8}} & e^{-\frac{i2\pi \cdot 6 \cdot 7}{8}} \\ 1 & e^{-\frac{i2\pi \cdot 7}{8}} & e^{-\frac{i2\pi \cdot 7 \cdot 2}{8}} & e^{-\frac{i2\pi \cdot 7 \cdot 3}{8}} & e^{-\frac{i2\pi \cdot 7 \cdot 4}{8}} & e^{-\frac{i2\pi \cdot 7 \cdot 5}{8}} & e^{-\frac{i2\pi \cdot 7 \cdot 6}{8}} & e^{-\frac{i2\pi \cdot 7 \cdot 7}{8}} \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix}$$

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Can we express this in terms of $F_{[4]}$?

$$F_{[4]} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & e^{-\frac{i2\pi}{4}} & e^{-\frac{i2\pi \cdot 2}{4}} & e^{-\frac{i2\pi \cdot 3}{4}} \\ 1 & e^{-\frac{i2\pi \cdot 2}{4}} & e^{-\frac{i2\pi \cdot 2 \cdot 2}{4}} & e^{-\frac{i2\pi \cdot 2 \cdot 3}{4}} \\ 1 & e^{-\frac{i2\pi \cdot 3}{4}} & e^{-\frac{i2\pi \cdot 3 \cdot 2}{4}} & e^{-\frac{i2\pi \cdot 3 \cdot 3}{4}} \end{bmatrix}$$

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Can we express this in terms of $F_{[4]}$?

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$$\begin{bmatrix} y_4 \\ y_5 \\ y_6 \\ y_7 \end{bmatrix} = \begin{bmatrix} 1 & e^{-\frac{i2\pi \cdot 4 \cdot 2}{8}} & e^{-\frac{i2\pi \cdot 4 \cdot 4}{8}} & e^{-\frac{i2\pi \cdot 4 \cdot 6}{8}} \\ 1 & e^{-\frac{i2\pi \cdot 5 \cdot 2}{8}} & e^{-\frac{i2\pi \cdot 5 \cdot 4}{8}} & e^{-\frac{i2\pi \cdot 5 \cdot 6}{8}} \\ 1 & e^{-\frac{i2\pi \cdot 6 \cdot 2}{8}} & e^{-\frac{i2\pi \cdot 6 \cdot 4}{8}} & e^{-\frac{i2\pi \cdot 6 \cdot 6}{8}} \\ 1 & e^{-\frac{i2\pi \cdot 7 \cdot 2}{8}} & e^{-\frac{i2\pi \cdot 7 \cdot 4}{8}} & e^{-\frac{i2\pi \cdot 7 \cdot 6}{8}} \end{bmatrix} \begin{bmatrix} x_0 \\ x_2 \\ x_4 \\ x_6 \end{bmatrix} + \begin{bmatrix} e^{-\frac{i2\pi \cdot 4}{8}} & e^{-\frac{i2\pi \cdot 4 \cdot 3}{8}} & e^{-\frac{i2\pi \cdot 4 \cdot 5}{8}} & e^{-\frac{i2\pi \cdot 4 \cdot 7}{8}} \\ e^{-\frac{i2\pi \cdot 5}{8}} & e^{-\frac{i2\pi \cdot 5 \cdot 3}{8}} & e^{-\frac{i2\pi \cdot 5 \cdot 5}{8}} & e^{-\frac{i2\pi \cdot 5 \cdot 7}{8}} \\ e^{-\frac{i2\pi \cdot 6}{8}} & e^{-\frac{i2\pi \cdot 6 \cdot 3}{8}} & e^{-\frac{i2\pi \cdot 6 \cdot 5}{8}} & e^{-\frac{i2\pi \cdot 6 \cdot 7}{8}} \\ e^{-\frac{i2\pi \cdot 7}{8}} & e^{-\frac{i2\pi \cdot 7 \cdot 3}{8}} & e^{-\frac{i2\pi \cdot 7 \cdot 5}{8}} & e^{-\frac{i2\pi \cdot 7 \cdot 7}{8}} \end{bmatrix} \begin{bmatrix} x_1 \\ x_3 \\ x_5 \\ x_7 \end{bmatrix}$$

$$\begin{bmatrix} y_4 \\ y_5 \\ y_6 \\ y_7 \end{bmatrix} =$$

Can we express this in terms of $F_{[4]}$?

$$F_{[4]} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & e^{-\frac{i2\pi}{4}} & e^{-\frac{i2\pi \cdot 2}{4}} & e^{-\frac{i2\pi \cdot 3}{4}} \\ 1 & e^{-\frac{i2\pi \cdot 2}{4}} & e^{-\frac{i2\pi \cdot 2 \cdot 2}{4}} & e^{-\frac{i2\pi \cdot 2 \cdot 3}{4}} \\ 1 & e^{-\frac{i2\pi \cdot 3}{4}} & e^{-\frac{i2\pi \cdot 3 \cdot 2}{4}} & e^{-\frac{i2\pi \cdot 3 \cdot 3}{4}} \end{bmatrix}$$

$$\begin{bmatrix} y_4 \\ y_5 \\ y_6 \\ y_7 \end{bmatrix} = \begin{bmatrix} 1 & e^{-\frac{i2\pi \cdot 4 \cdot 2}{8}} & e^{-\frac{i2\pi \cdot 4 \cdot 4}{8}} & e^{-\frac{i2\pi \cdot 4 \cdot 6}{8}} \\ 1 & e^{-\frac{i2\pi \cdot 5 \cdot 2}{8}} & e^{-\frac{i2\pi \cdot 5 \cdot 4}{8}} & e^{-\frac{i2\pi \cdot 5 \cdot 6}{8}} \\ 1 & e^{-\frac{i2\pi \cdot 6 \cdot 2}{8}} & e^{-\frac{i2\pi \cdot 6 \cdot 4}{8}} & e^{-\frac{i2\pi \cdot 6 \cdot 6}{8}} \\ 1 & e^{-\frac{i2\pi \cdot 7 \cdot 2}{8}} & e^{-\frac{i2\pi \cdot 7 \cdot 4}{8}} & e^{-\frac{i2\pi \cdot 7 \cdot 6}{8}} \end{bmatrix} \begin{bmatrix} x_0 \\ x_2 \\ x_4 \\ x_6 \end{bmatrix} + \begin{bmatrix} e^{-\frac{i2\pi \cdot 4}{8}} & e^{-\frac{i2\pi \cdot 4 \cdot 3}{8}} & e^{-\frac{i2\pi \cdot 4 \cdot 5}{8}} & e^{-\frac{i2\pi \cdot 4 \cdot 7}{8}} \\ e^{-\frac{i2\pi \cdot 5}{8}} & e^{-\frac{i2\pi \cdot 5 \cdot 3}{8}} & e^{-\frac{i2\pi \cdot 5 \cdot 5}{8}} & e^{-\frac{i2\pi \cdot 5 \cdot 7}{8}} \\ e^{-\frac{i2\pi \cdot 6}{8}} & e^{-\frac{i2\pi \cdot 6 \cdot 3}{8}} & e^{-\frac{i2\pi \cdot 6 \cdot 5}{8}} & e^{-\frac{i2\pi \cdot 6 \cdot 7}{8}} \\ e^{-\frac{i2\pi \cdot 7}{8}} & e^{-\frac{i2\pi \cdot 7 \cdot 3}{8}} & e^{-\frac{i2\pi \cdot 7 \cdot 5}{8}} & e^{-\frac{i2\pi \cdot 7 \cdot 7}{8}} \end{bmatrix} \begin{bmatrix} x_1 \\ x_3 \\ x_5 \\ x_7 \end{bmatrix}$$

$$\begin{bmatrix} y_4 \\ y_5 \\ y_6 \\ y_7 \end{bmatrix} = F_{[4]} \begin{bmatrix} x_0 \\ x_2 \\ x_4 \\ x_6 \end{bmatrix} + \begin{bmatrix} e^{-\frac{i2\pi \cdot 4}{8}} & 0 & 0 & 0 \\ 0 & e^{-\frac{i2\pi \cdot 5}{8}} & 0 & 0 \\ 0 & 0 & e^{-\frac{i2\pi \cdot 6}{8}} & 0 \\ 0 & 0 & 0 & e^{-\frac{i2\pi \cdot 7}{8}} \end{bmatrix} F_{[4]} \begin{bmatrix} x_1 \\ x_3 \\ x_5 \\ x_7 \end{bmatrix}$$

Even columns can be scaled to yield odd columns

$$\begin{bmatrix} e^{-2\pi i(0)/8} \\ e^{-2\pi i(1)/8} \\ e^{-2\pi i(2)/8} \\ e^{-2\pi i(3)/8} \\ \\ e^{-2\pi i(4)/8} \\ e^{-2\pi i(5)/8} \\ e^{-2\pi i(6)/8} \\ e^{-2\pi i(7)/8} \end{bmatrix} = \begin{bmatrix} \text{dark blue vertical bar} \\ \\ \text{light blue vertical bar} \end{bmatrix} = \begin{bmatrix} \text{dark green vertical bar} \\ \\ \text{light green vertical bar} \end{bmatrix}$$

Top even submatrix and bottom even submatrix are both an $\frac{N}{2} \times \frac{N}{2}$ DFT matrix

$$\begin{bmatrix} \text{---} & \text{---} & \text{---} & \text{---} \\ \text{---} & \text{---} & \text{---} & \text{---} \\ \text{---} & \text{---} & \text{---} & \text{---} \\ \text{---} & \text{---} & \text{---} & \text{---} \end{bmatrix} = \begin{bmatrix} \text{---} & \text{---} & \text{---} & \text{---} \\ \text{---} & \text{---} & \text{---} & \text{---} \\ \text{---} & \text{---} & \text{---} & \text{---} \\ \text{---} & \text{---} & \text{---} & \text{---} \end{bmatrix}$$

FFT identity

$$\begin{bmatrix} \hat{x}[0] \\ \hat{x}[1] \\ \hat{x}[2] \\ \hat{x}[3] \end{bmatrix} = \begin{bmatrix} \text{ } \\ \text{ } \\ \text{ } \\ \text{ } \end{bmatrix} + \begin{bmatrix} \vec{x}[0] \\ \vec{x}[2] \\ \vec{x}[4] \\ \vec{x}[6] \end{bmatrix} + \begin{bmatrix} e^{-2\pi i(0)/8} \\ e^{-2\pi i(1)/8} \\ e^{-2\pi i(2)/8} \\ e^{-2\pi i(3)/8} \end{bmatrix} \begin{bmatrix} \text{ } \\ \text{ } \\ \text{ } \\ \text{ } \end{bmatrix} \begin{bmatrix} \vec{x}[1] \\ \vec{x}[3] \\ \vec{x}[5] \\ \vec{x}[7] \end{bmatrix}$$
$$\begin{bmatrix} \hat{x}[4] \\ \hat{x}[5] \\ \hat{x}[6] \\ \hat{x}[7] \end{bmatrix} = \begin{bmatrix} \text{ } \\ \text{ } \\ \text{ } \\ \text{ } \end{bmatrix} + \begin{bmatrix} \vec{x}[0] \\ \vec{x}[2] \\ \vec{x}[4] \\ \vec{x}[6] \end{bmatrix} + \begin{bmatrix} e^{-2\pi i(4)/8} \\ e^{-2\pi i(5)/8} \\ e^{-2\pi i(6)/8} \\ e^{-2\pi i(7)/8} \end{bmatrix} \begin{bmatrix} \text{ } \\ \text{ } \\ \text{ } \\ \text{ } \end{bmatrix} \begin{bmatrix} \vec{x}[1] \\ \vec{x}[3] \\ \vec{x}[5] \\ \vec{x}[7] \end{bmatrix}$$

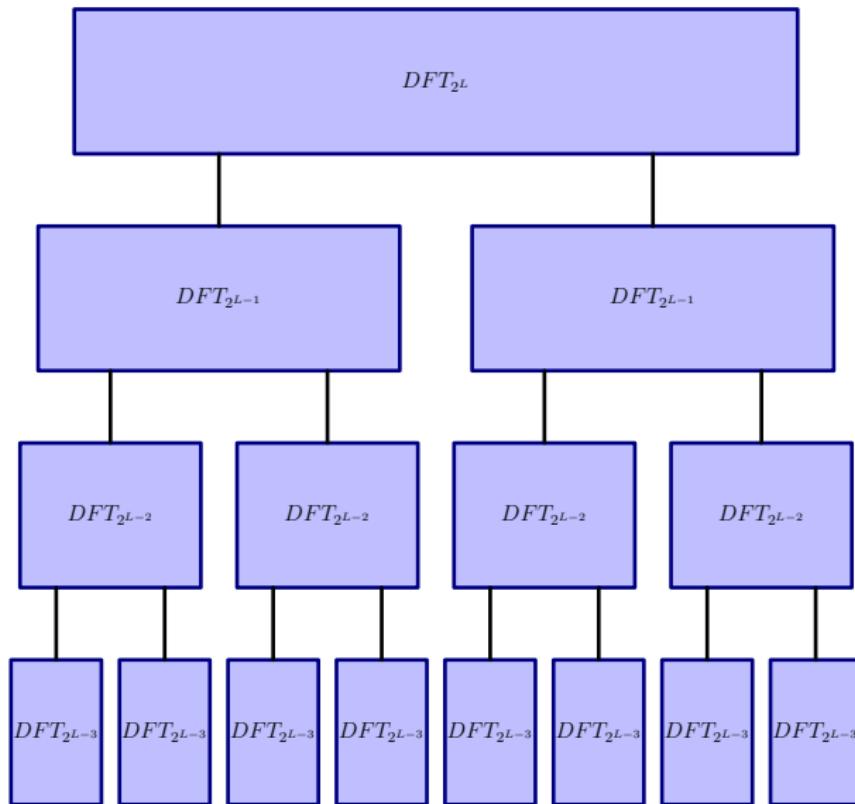
Cooley-Tukey Fast Fourier transform

1. Compute $F_{[N/2]}x_{\text{even}}$.
2. Compute $F_{[N/2]}x_{\text{odd}}$.
3. For $k = 0, 1, \dots, N/2 - 1$ set

$$F_N x [k] := F_{[N/2]}x_{\text{even}} [k] + \exp \left(-\frac{i2\pi k}{N} \right) F_{[N/2]}x_{\text{odd}} [k]$$

$$F_N x [k + N/2] := F_{[N/2]}x_{\text{even}} [k] - \exp \left(-\frac{i2\pi k}{N} \right) F_{[N/2]}x_{\text{odd}} [k]$$

Complexity



Complexity

Assume $N = 2^L$

$L = \log_2 N$ levels

At level $m \in \{1, \dots, L\}$ there are 2^m nodes

At each node, scale a vector of dim 2^{L-m} and add to another vector

Complexity at each node:

Complexity at each level:

Complexity

Assume $N = 2^L$

$L = \log_2 N$ levels

At level $m \in \{1, \dots, L\}$ there are 2^m nodes

At each node, scale a vector of dim 2^{L-m} and add to another vector

Complexity at each node: 2^{L-m}

Complexity at each level:

Complexity

Assume $N = 2^L$

$L = \log_2 N$ levels

At level $m \in \{1, \dots, L\}$ there are 2^m nodes

At each node, scale a vector of dim 2^{L-m} and add to another vector

Complexity at each node: 2^{L-m}

Complexity at each level: $2^{L-m}2^m = 2^L = N$

Complexity

Assume $N = 2^L$

$L = \log_2 N$ levels

At level $m \in \{1, \dots, L\}$ there are 2^m nodes

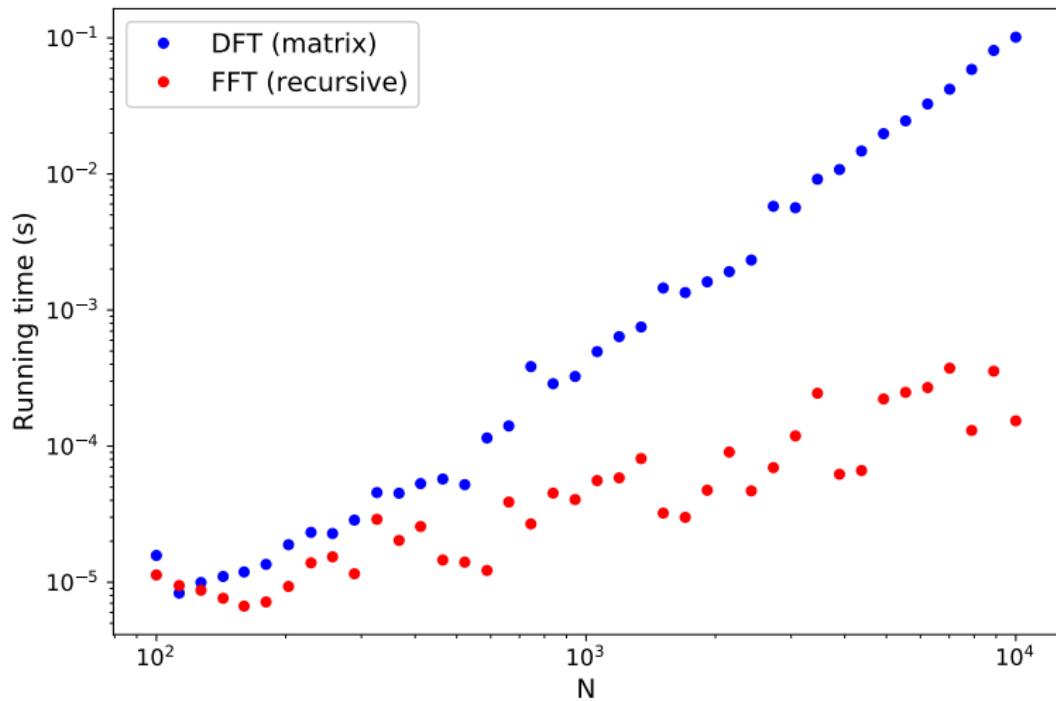
At each node, scale a vector of dim 2^{L-m} and add to another vector

Complexity at each node: 2^{L-m}

Complexity at each level: $2^{L-m}2^m = 2^L = N$

Complexity is $O(N \log N)$!

In practice



The fast Fourier transform

2D Fourier series

2D complex sinusoids

Family of complex sinusoids on $[0, T) \times [0, T)$

$$\phi_{k_1, k_2}^{\text{2D}}(t_1, t_2) := \exp\left(\frac{i2\pi k_1 t_1}{T}\right) \exp\left(\frac{i2\pi k_2 t_2}{T}\right), \quad k_1, k_2 \in \mathbb{Z}$$

2D Fourier series

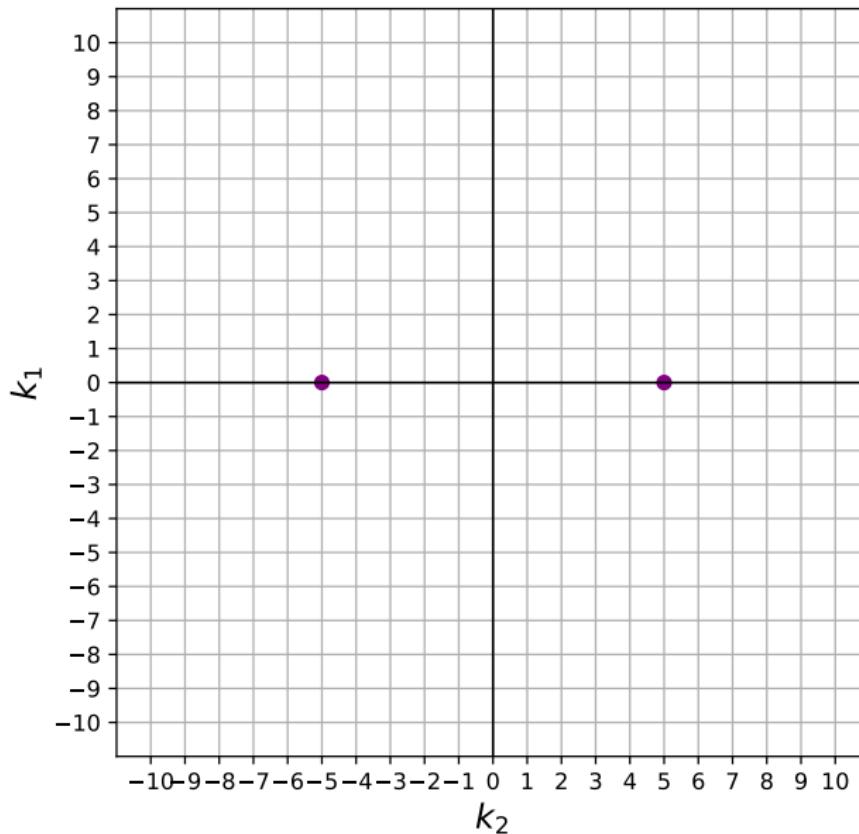
Fourier series coefficients of a function $x \in \mathcal{L}_2 [0, T]$

$$\begin{aligned}\hat{x}[k_1, k_2] &:= \left\langle x, \phi_{k_1, k_2}^{2D} \right\rangle \\ &= \int_{t_1=0}^T \int_{t_2=0}^T x(t_1, t_2) \exp\left(-\frac{i2\pi k_1 t_1}{T}\right) \exp\left(-\frac{i2\pi k_2 t_2}{T}\right) dt_1 dt_2\end{aligned}$$

The Fourier series of order k_{c1}, k_{c2} is defined as

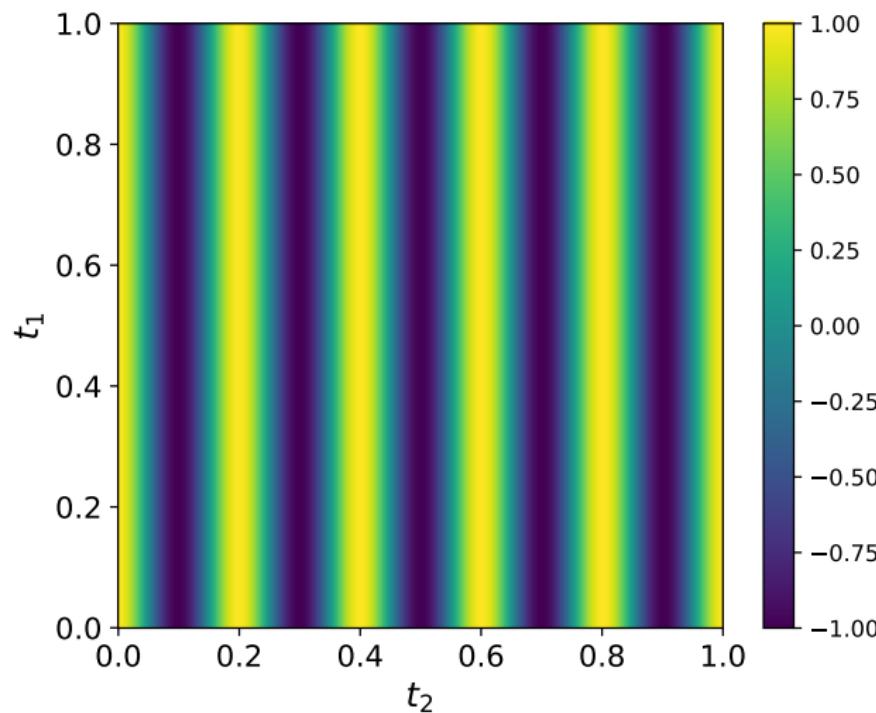
$$\mathcal{F}_{k_{c1}, k_{c2}} \{x\} := \frac{1}{T^2} \sum_{k_1=-k_{c1}}^{k_{c1}} \sum_{k_2=-k_{c2}}^{k_{c2}} \hat{x}[k_1, k_2] \phi_{k_1, k_2}^{2D}.$$

$$\frac{\phi_{0,5}^{2D} + \phi_{0,-5}^{2D}}{2}$$

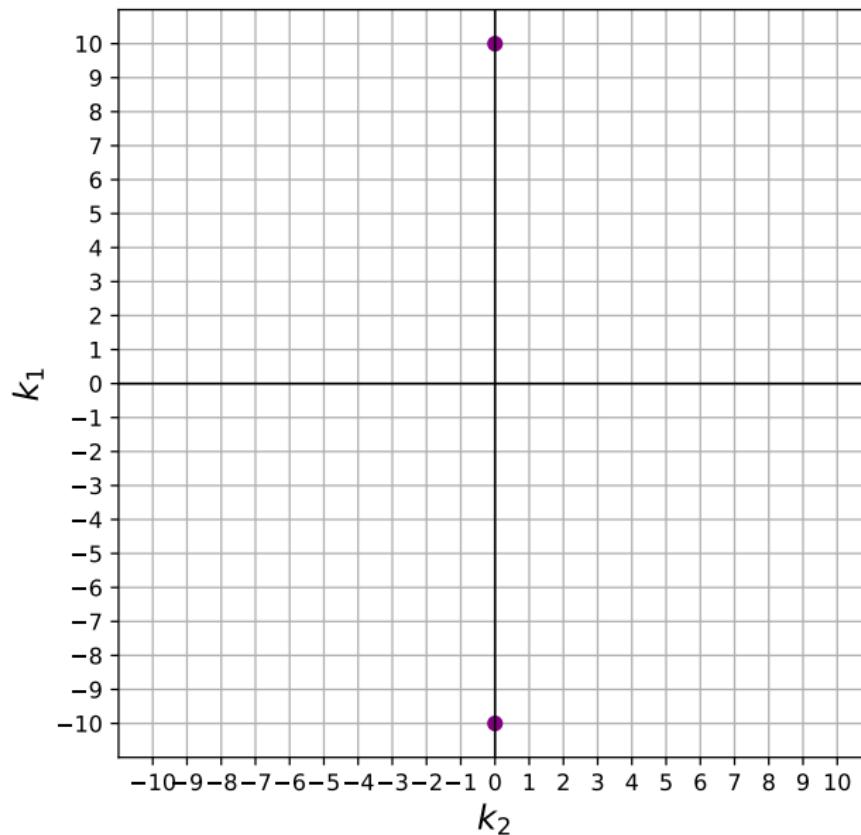


$$\frac{\phi_{0,5}^{\text{2D}}+\phi_{0,-5}^{\text{2D}}}{2}$$

$$\frac{\phi_{0,5}^{2D} + \phi_{0,-5}^{2D}}{2}$$

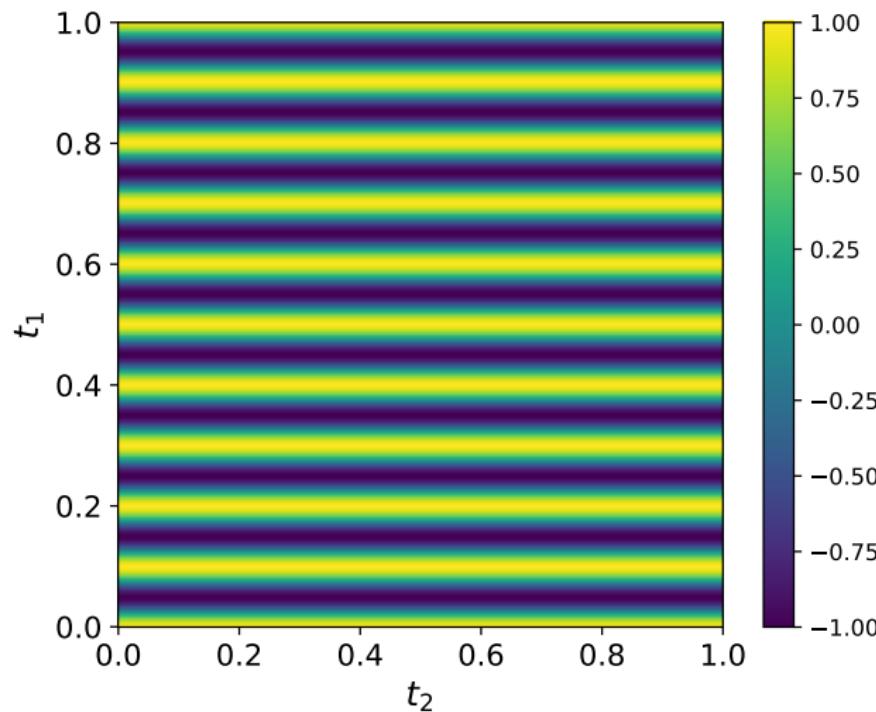


$$\frac{\phi_{10,0}^{2D} + \phi_{-10,0}^{2D}}{2}$$

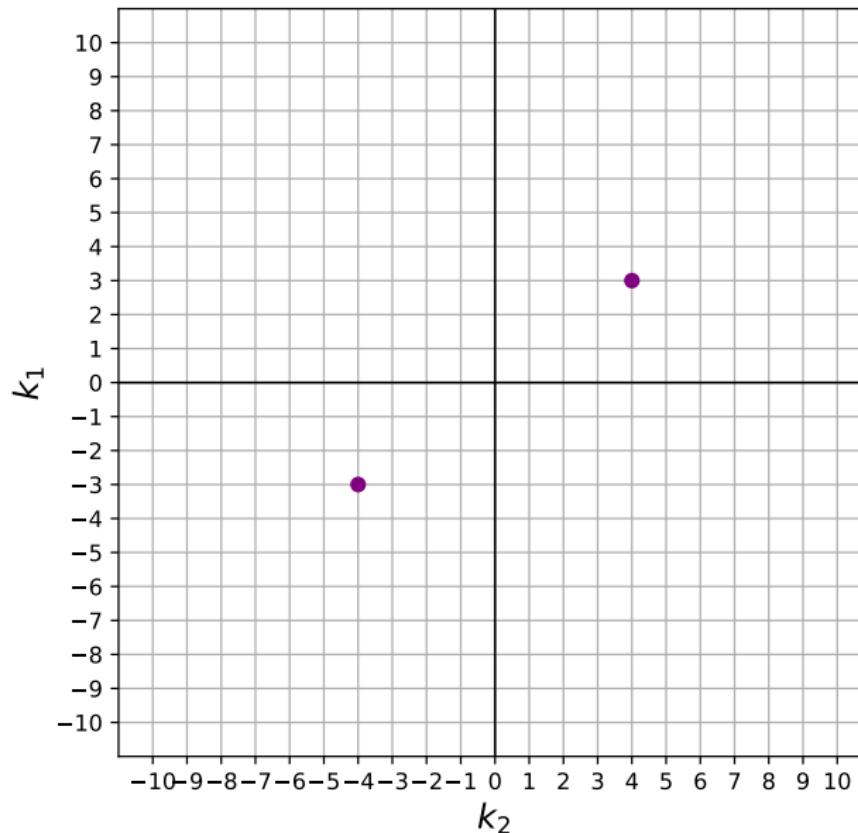


$$\frac{\phi_{10,0}^{\text{2D}}+\phi_{-10,0}^{\text{2D}}}{2}$$

$$\frac{\phi_{10,0}^{2D} + \phi_{-10,0}^{2D}}{2}$$

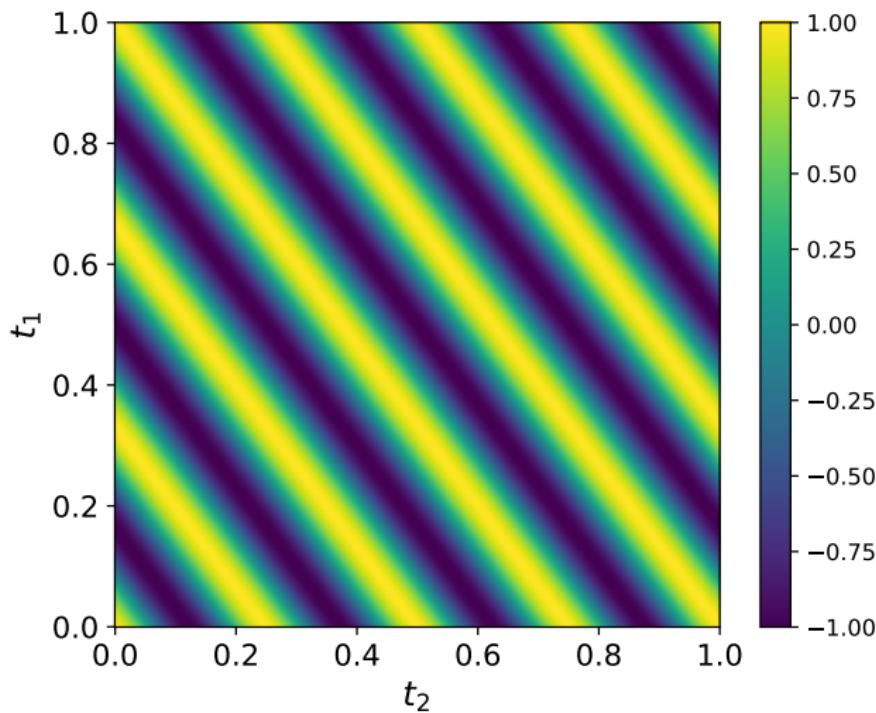


$$\frac{\phi_{3,4}^{2D} + \phi_{-3,-4}^{2D}}{2}$$

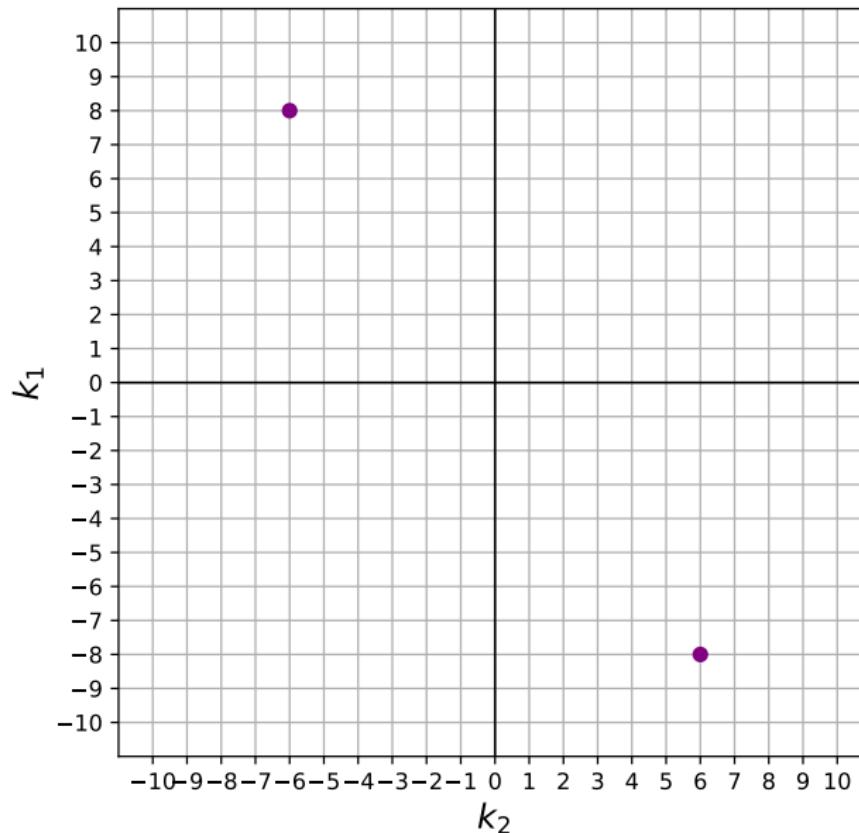


$$\frac{\phi_{3,4}^{\text{2D}}+\phi_{-3,-4}^{\text{2D}}}{2}$$

$$\frac{\phi_{3,4}^{2D} + \phi_{-3,-4}^{2D}}{2}$$

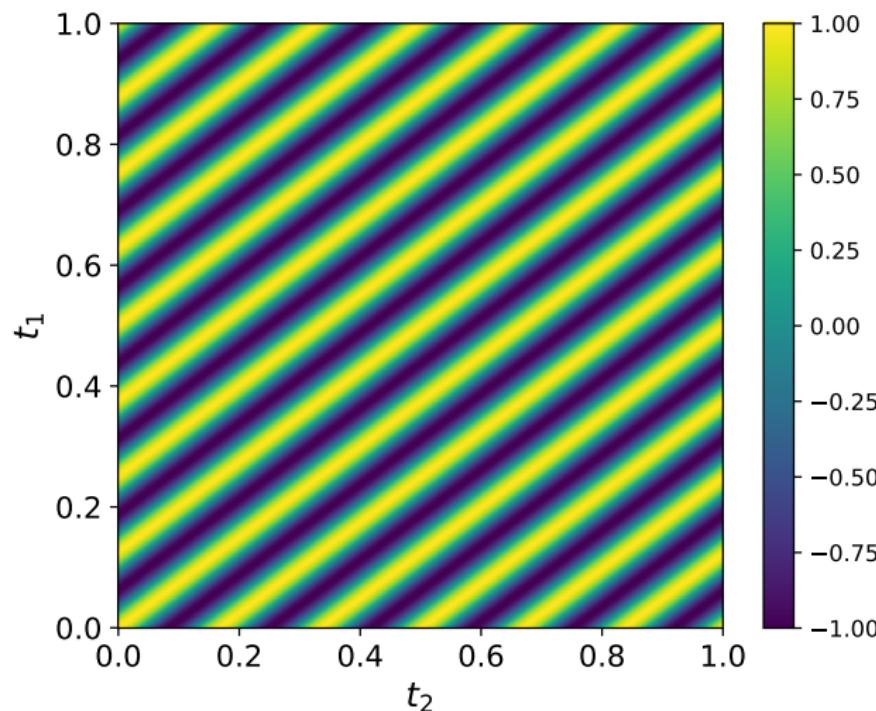


$$\frac{\phi_{8,-6}^{2D} + \phi_{-8,6}^{2D}}{2}$$



$$\frac{\phi_{8,-6}^{\text{2D}}+\phi_{-8,6}^{\text{2D}}}{2}$$

$$\frac{\phi_{8,-6}^{2D} + \phi_{-8,6}^{2D}}{2}$$



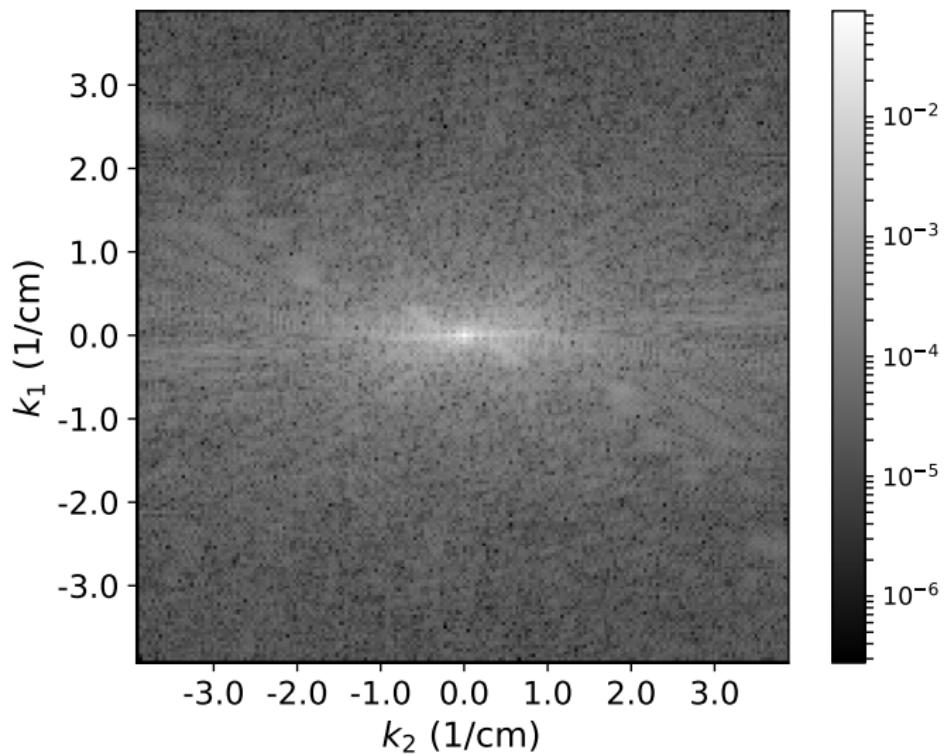
Magnetic resonance imaging

Non-invasive medical-imaging technique

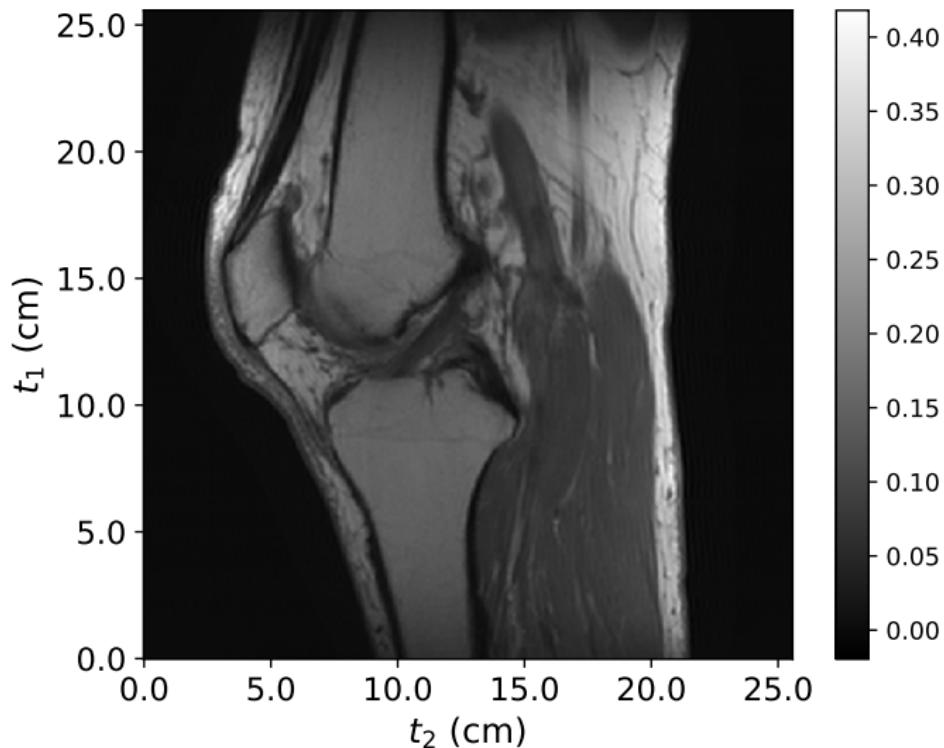
Measures response of atomic nuclei in biological tissues to high-frequency radio waves when placed in a strong magnetic field

Radio waves adjusted so that each measurement equals 2D Fourier coefficients of proton density of hydrogen atoms in a region of interest

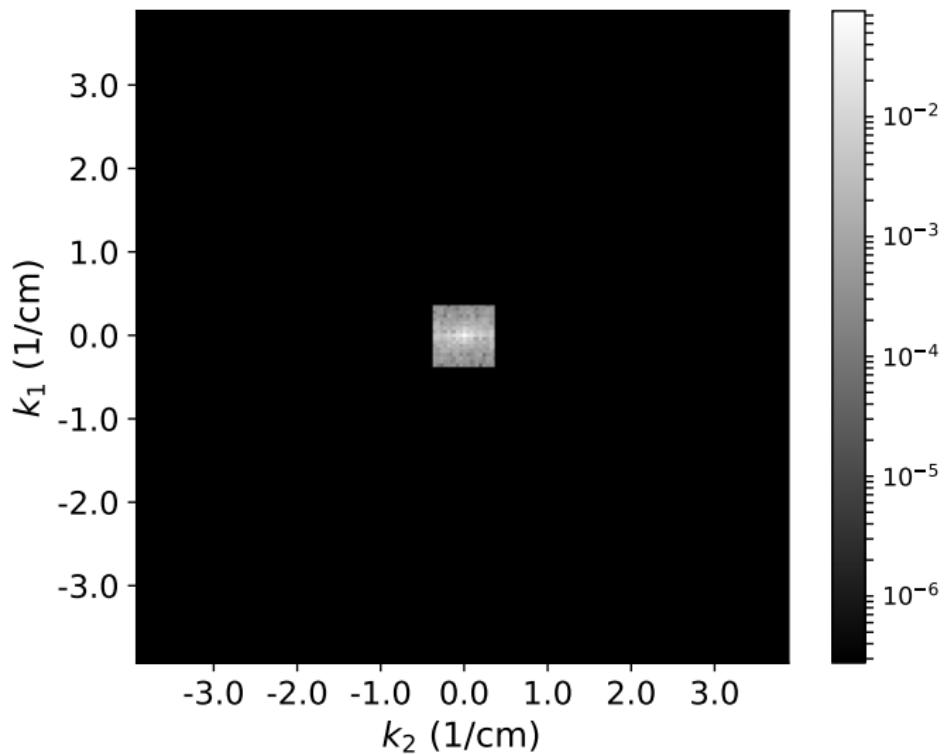
Data



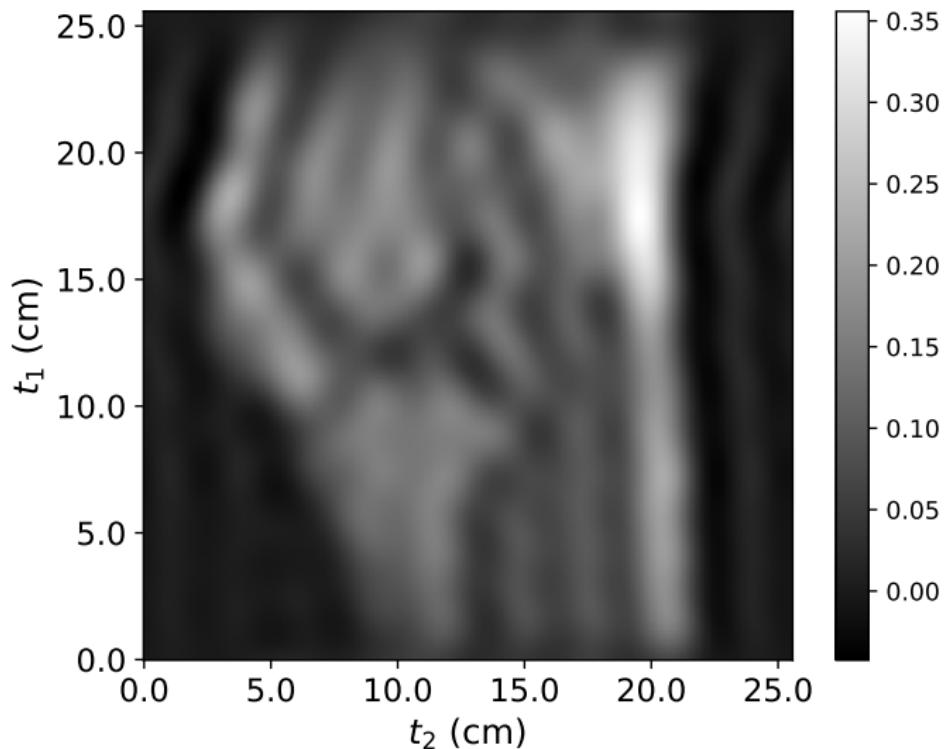
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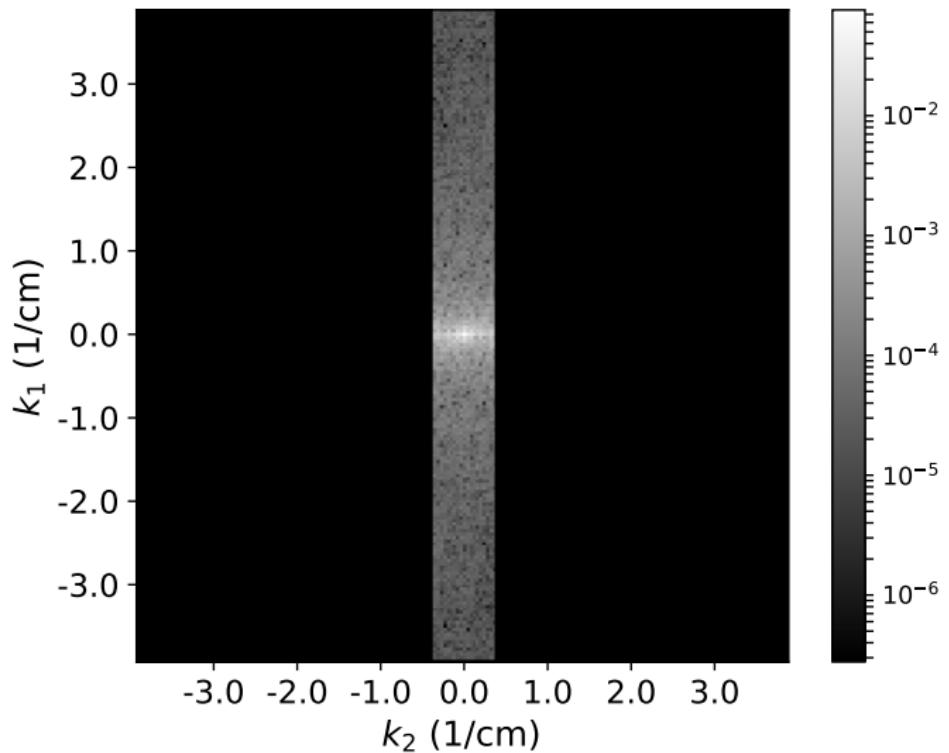
Data



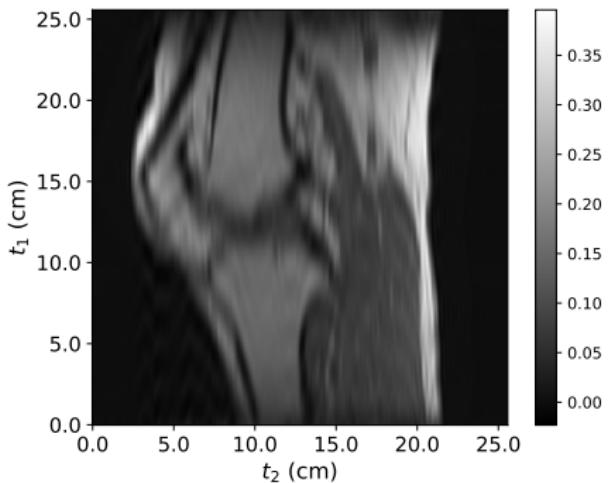
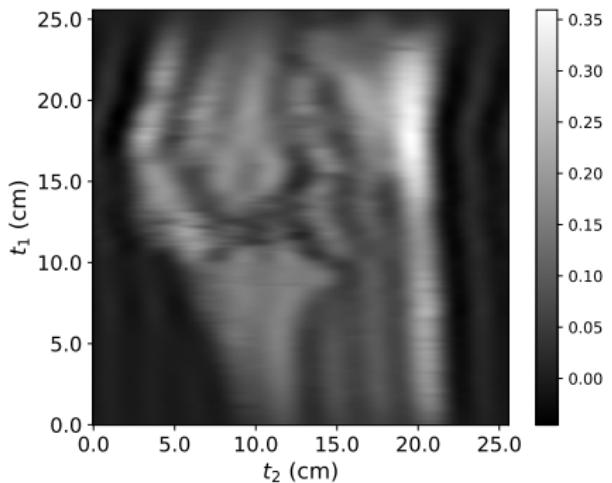
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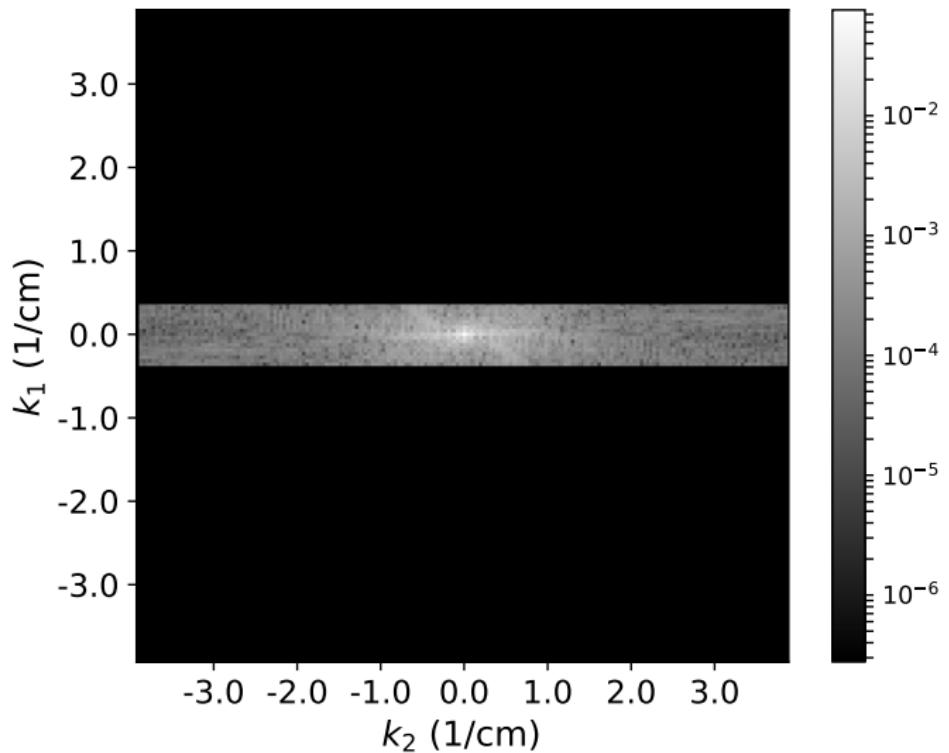
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