Fourier Series

DS-GA 1013 / MATH-GA 2824 Mathematical Tools for Data Science

Carlos Fernandez-Granda
Prerequisites

Calculus (complex numbers)

Linear algebra (orthogonality, basis, projections)
Signal processing

Signal: any structured object of interest (images, audio, video, etc.)

Modeled as function of space, time, etc.

Finding adequate representations is crucial to process signals effectively
Electrocardiogram
Signals as functions

We model signals as functions on an interval \([a, b] \subset \mathbb{R}\).

Inner product:

\[
\langle x, y \rangle := \int_{a}^{b} x(t) \overline{y(t)} \, dt
\]
Sinusoids

Functions that oscillate at a certain frequency:

\[ a \cos(2\pi ft + \theta) \]

- **Amplitude**: \( a \)
- **Frequency**: \( f \)
- **Time index**: \( t \)
- **Phase**: \( \theta \)

\[
a \cos \left( 2\pi f \left( t + \frac{1}{f} \right) + \theta \right) = a \cos (2\pi ft + \theta + 2\pi) = a \cos (2\pi ft + \theta)
\]
Problem

How to represent sinusoids with same frequency but different phases?
The complex sinusoid with frequency \( f \in \mathbb{R} \) is given by

\[
\exp(i2\pi ft) := \cos(2\pi ft) + i \sin(2\pi ft)
\]
Complex sinusoid
Complex sinusoids

We can express any real sinusoid in terms of complex sinusoids

\[ \exp(i2\pi ft + i\theta) = \cos(2\pi ft + \theta) + i \sin(2\pi ft + \theta) \]

\[ \exp(-i2\pi ft - i\theta) = \cos(-2\pi ft - \theta) + i \sin(-2\pi ft - \theta) \]

\[ = \cos(2\pi ft + \theta) - i \sin(2\pi ft + \theta) \]

\[ \cos(2\pi ft + \theta) = \frac{\exp(i2\pi ft + i\theta) + \exp(-i2\pi ft - i\theta)}{2} \]

\[ = \frac{\exp(i\theta)}{2} \exp(i2\pi ft) + \frac{\exp(-i\theta)}{2} \exp(-i2\pi ft) \]

The phase is encoded in the complex amplitude!

Linear subspace spanned by \( \exp(i2\pi ft) \) and \( \exp(-i2\pi ft) \) contains all real sinusoids with frequency \( f \)
Family of complex sinusoids on $[0, T]$ 

$$\phi_k(t) := \exp\left(\frac{i2\pi kt}{T}\right), \quad k \in \mathbb{Z},$$

$$||\phi_j|| = \sqrt{T}$$

If $j \neq k$, $\phi_j$ and $\phi_k$ are orthogonal.
Proof

\[ \langle \phi_k, \phi_j \rangle = \int_0^T \phi_k(t) \overline{\phi_j(t)} \, dt \]

\[ = \int_0^T \exp \left( \frac{i2\pi (k) t}{T} \right) \exp \left( \frac{i2\pi (-j) t}{T} \right) \, dt \]

\[ = \int_0^T \exp \left( \frac{i2\pi (k-j) t}{T} \right) \, dt \]

\[ = \begin{cases} 
\int_0^T dt = T & \text{if } k = j \\
\frac{T}{i2\pi(k-j)} \left( \exp \left( i2\pi (k-j) \right) - 1 \right) = 0 & \text{if } k \neq j 
\end{cases} \]
Orthogonal basis

The best approximation of a vector $x$ as a linear combination of $n$ orthonormal vectors $v_1, \ldots, v_n$ is

$$\sum_{j=1}^{n} \langle x, v_j \rangle v_j = \arg \min_{y \in \text{span}\{v_1, \ldots, v_n\}} \|x - y\|_2^2$$
Fourier series as a projection

The best approximation of a signal $x$ as a linear combination of $n$ complex sinusoids $\phi_1, \ldots, \phi_n$ is

$$\sum_{k=1}^{n} \left\langle x, \frac{1}{\sqrt{T}} \phi_k \right\rangle \frac{1}{\sqrt{T}} \phi_k = \frac{1}{T} \sum_{k=1}^{n} \langle x, \phi_k \rangle \phi_k$$
Fourier series

The Fourier series coefficients of \( x \in L_2 [0, T] \) are

\[
\hat{x}[k] := \langle x, \phi_k \rangle = \int_0^T x(t) \exp \left( - \frac{i2\pi kt}{T} \right) \, dt
\]

The Fourier series of order \( k_c \) is defined as

\[
\mathcal{F}_{k_c} \{ x \} := \frac{1}{T} \sum_{k=-k_c}^{k_c} \hat{x}[k] \phi_k
\]

The Fourier series of \( x \) is \( \lim_{k_c \to \infty} \mathcal{F}_{k_c} \{ x \} \)
Convergence of Fourier series

For any function $x \in \mathcal{L}_2[0, T)$

$$\lim_{k \to \infty} \|x - \mathcal{F}_k \{x\}\|_{\mathcal{L}_2} = 0$$
Electrocardiogram
Electrocardiogram

\[ \hat{x}[k] := \langle x, \phi_k \rangle = \int_0^T x(t) \exp \left( -\frac{i2\pi kt}{T} \right) \, dt \]

\[ = \vert \hat{x}[k] \vert \exp (\xi_k) \]
Electrocardiogram: Fourier coefficients (magnitude)
Real valued signals

Let $\hat{x}[k] = \alpha_k \exp(i\xi_k)$ be the Fourier series coefficients of a real-valued function $x \in L_2[0, T]$

$$\hat{x}[-k] := \int_0^T x(t) \exp\left(\frac{i2\pi kt}{T}\right) \, dt$$

$$= \int_0^T x(t) \exp\left(-\frac{i2\pi kt}{T}\right) \, dt \quad \text{because } x \text{ is real valued}$$

$$= \overline{\hat{x}[k]}$$

$$= \alpha_k \exp(-i\xi_k)$$
Real valued signals

Component corresponding to frequency $\frac{k}{T}$

$$\hat{x}[k]\phi_k + \hat{x}[-k]\phi_{-k}$$

$$= \alpha_k \exp(i\xi_k) \exp\left(\frac{i2\pi kt}{T}\right) + \alpha_k \exp(-i\xi_k) \exp\left(-\frac{i2\pi kt}{T}\right)$$

$$= \alpha_k \left(\exp\left(\frac{i2\pi kt}{T} + \xi_k\right) + \exp\left(-\frac{i2\pi kt}{T} - \xi_k\right)\right)$$

$$= 2\alpha_k \cos\left(\frac{2\pi kt}{T} + \xi_k\right)$$

$$\mathcal{F}_{k_c}\{x\} := \frac{1}{T} \sum_{k=-k_c}^{k_c} \hat{x}[k]\phi_k = \alpha_0 + \frac{1}{T} \sum_{k=0}^{k_c} 2\alpha_k \cos\left(\frac{2\pi kt}{T} + \xi_k\right)$$
Fourier components
Fourier series

![Fourier series graph](image-url)
Fourier components
Fourier series
Electrocardiogram data
Electrocardiogram features

- P wave
- PR segment
- PR interval
- QRS complex
- ST segment
- QT interval
Problem: Baseline wandering
Electrocardiogram: Fourier coefficients (magnitude)
Filtering

**Idea:** Process signal by removing (or attenuating) frequency components

High-pass filtering to correct baseline wandering

\[
x \approx \frac{1}{T} \sum_{k=-k_{\text{max}}}^{-k_{\text{max}}} \hat{x}[k] \phi_k
\]

\[
x_{\text{high-pass}} := \frac{1}{T} \sum_{k=-k_{\text{max}}}^{-k_{\text{thresh}}} \hat{x}[k] \phi_k + \frac{1}{T} \sum_{k=k_{\text{thresh}}}^{k_{\text{max}}} \hat{x}[k] \phi_k
\]
Electrocardiogram after high-pass filtering
Electrocardiogram after high-pass filtering
Problem: Electric-grid interference
Fourier coefficients (magnitude)
Idea: Can we remove the interference by filtering?

Band-stop filtering

\[ x_{\text{filtered}} := \frac{1}{T} \sum_{k=-k_{\text{max}}}^{-k_{\text{band-end}}} \hat{x}[k] \phi_k + \frac{1}{T} \sum_{k=-k_{\text{thresh}}}^{-k_{\text{thresh}}} \hat{x}[k] \phi_k \]

\[ + \frac{1}{T} \sum_{k=k_{\text{thresh}}}^{k_{\text{band-ini}}} \hat{x}[k] \phi_k + \frac{1}{T} \sum_{k=k_{\text{band-ini}}}^{k_{\text{max}}} \hat{x}[k] \phi_k \]
Filtered electrocardiogram
Filtered electrocardiogram
Electrocardiogram features
What have we learned

Complex exponentials with different frequencies form an orthogonal basis.

This provides a representation of signals in terms of sinusoids.

We can use this representation to filter different components of signals.