Fourier Series

DS-GA 1013 / MATH-GA 2824 Mathematical Tools for Data Science

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Prerequisites

Calculus (complex numbers)

Linear algebra (orthogonality, basis, projections)

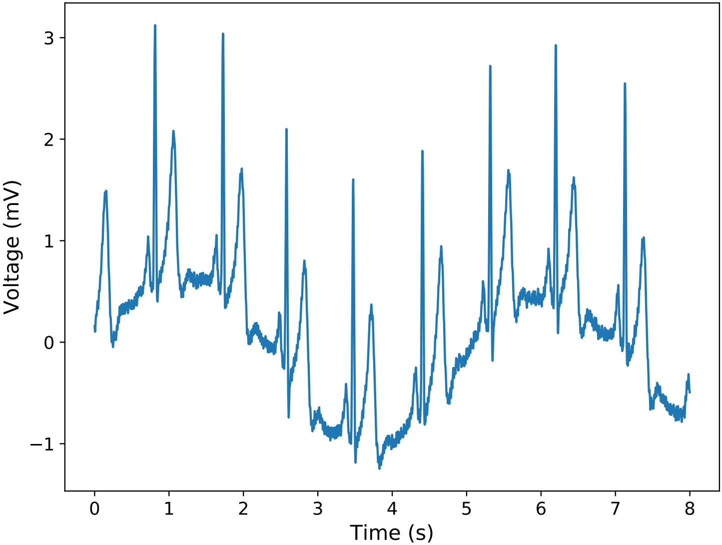
Signal processing

Signal: any structured object of interest (images, audio, video, etc.)

Modeled as function of space, time, etc.

Finding adequate representations is crucial to process signals effectively

Electrocardiogram



Signals as functions

We model signals as functions on an interval $[a, b] \subset \mathbb{R}$

Inner product:

$$\langle x, y \rangle := \int_a^b x(t)\,\overline{y(t)}\,\mathrm{d}t$$

Sinusoids

Functions that oscillate at a certain frequency:

$$a \cos(2\pi f t + \theta)$$

- Amplitude: a
- Frequency: f
- Time index: t
- Phase: θ

$$\begin{aligned} a \cos\left(2\pi f \left(t + \frac{1}{f}\right) + \theta\right) &= a \cos\left(2\pi f t + \theta + 2\pi\right) \\ &= a \cos\left(2\pi f t + \theta\right) \end{aligned}$$

Problem

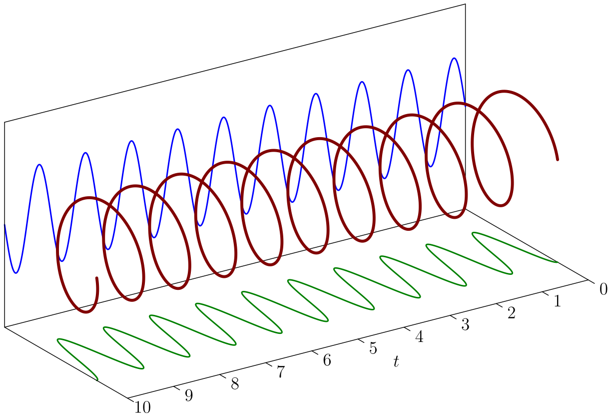
How to represent sinusoids with same frequency but different phases?

Complex sinusoids

The complex sinusoid with frequency $f \in \mathbb{R}$ is given by

$$\exp(i2\pi ft) := \cos(2\pi ft) + i\sin(2\pi ft)$$

Complex sinusoid



Complex sinusoids

We can express any real sinusoid in terms of complex sinusoids

$$\exp(i2\pi ft + i\theta) = \cos(2\pi ft + \theta) + i\sin(2\pi ft + \theta)$$

$$\begin{aligned}\exp(-i2\pi ft - i\theta) &= \cos(-2\pi ft - \theta) + i\sin(-2\pi ft - \theta)\\ &= \cos(2\pi ft + \theta) - i\sin(2\pi ft + \theta)\end{aligned}$$

$$\begin{aligned}\cos(2\pi ft + \theta) &= \frac{\exp(i2\pi ft + i\theta) + \exp(-i2\pi ft - i\theta)}{2}\\ &= \frac{\exp(i\theta)}{2}\exp(i2\pi ft) + \frac{\exp(-i\theta)}{2}\exp(-i2\pi ft)\end{aligned}$$

The phase is encoded in the complex amplitude!

Linear subspace spanned by $\exp(i2\pi ft)$ and $\exp(-i2\pi ft)$ contains all real sinusoids with frequency f

Family of complex sinusoids on $[0, T]$

$$\phi_k(t) := \exp\left(\frac{i2\pi kt}{T}\right), \qquad k \in \mathbb{Z},$$

$||\phi_j|| = \sqrt{T}$

If $j \neq k$, ϕ_j and ϕ_k are orthogonal

Proof

$$
\begin{aligned}
\langle \phi_k, \phi_j \rangle &= \int_0^T \phi_k(t)\,\overline{\phi_j(t)}\,\mathrm{d}t \\
&= \int_0^T \exp\left(\frac{i2\pi(k)\,t}{T}\right)\exp\left(\frac{i2\pi(-j)\,t}{T}\right)\,\mathrm{d}t \\
&= \int_0^T \exp\left(\frac{i2\pi(k-j)\,t}{T}\right)\,\mathrm{d}t \\
&= \begin{cases} \int_0^T \mathrm{d}t = T & \text{if } k = j \\ \frac{T}{i2\pi(k-j)}\left(\exp\left(i2\pi(k-j)\right) - 1\right) = 0 & \text{if } k \neq j \end{cases}
\end{aligned}
$$

Orthogonal basis

The best approximation of a vector x as a linear combination of n orthonormal vectors $v_1, \ldots, v_n$ is

$$\sum_{j=1}^{n} \langle x, v_j \rangle \, v_j = \arg \min_{y \in \text{span}\{v_1,\ldots,v_n\}} ||x - y||_2^2$$

Fourier series as a projection

The best approximation of a signal x as a linear combination of n complex sinusoids $\phi_1, \ldots, \phi_n$ is

$$\sum_{k=1}^{n} \left\langle x, \frac{1}{\sqrt{T}}\phi_k \right\rangle \frac{1}{\sqrt{T}}\phi_k = \frac{1}{T}\sum_{k=1}^{n} \langle x, \phi_k \rangle \phi_k$$

Fourier series

The Fourier series coefficients of $x \in \mathcal{L}_2\left[0, T\right]$ are

$$\hat{x}[k] := \langle x, \phi_k \rangle = \int_0^T x(t) \exp\left(-\frac{i2\pi kt}{T}\right) \mathrm{d}t$$

The Fourier series of order k_c is defined as

$$\mathcal{F}_{k_c}\left\{x\right\} := \frac{1}{T} \sum_{k=-k_c}^{k_c} \hat{x}[k]\phi_k$$

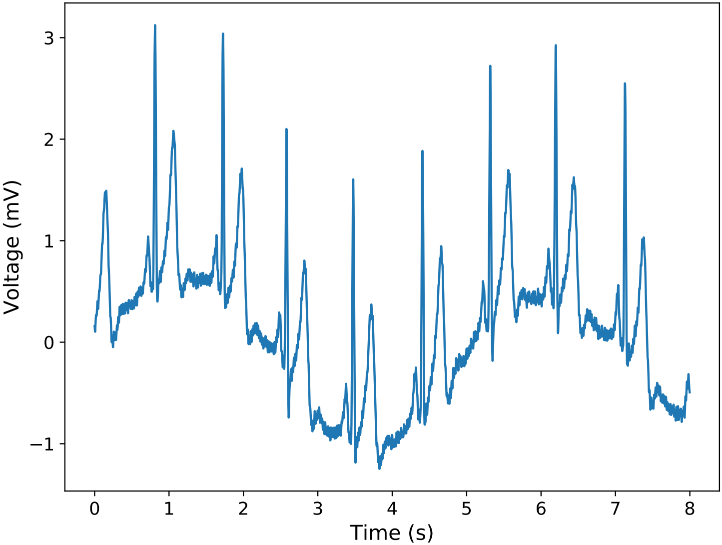
The Fourier series of x is $\lim_{k_c \to \infty} \mathcal{F}_{k_c}\left\{x\right\}$

Convergence of Fourier series

For any function $x \in \mathcal{L}_2[0, T)$

$$\lim_{k \to \infty} ||x - \mathcal{F}_k \{x\}||_{\mathcal{L}_2} = 0$$

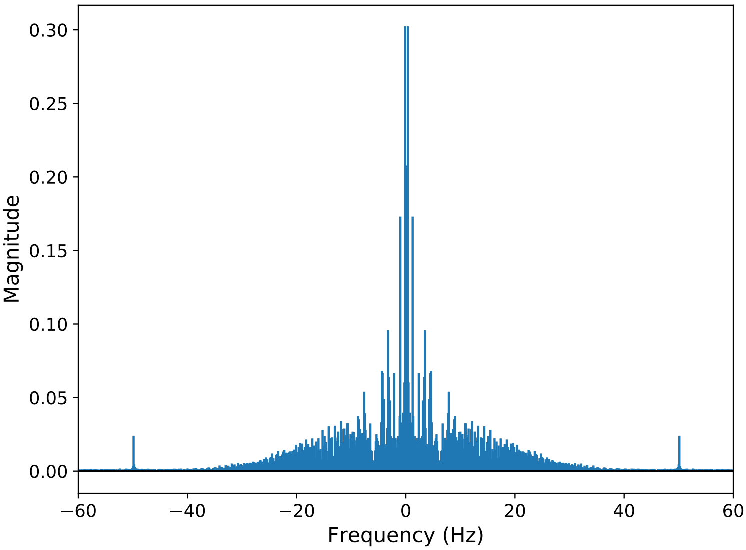
Electrocardiogram



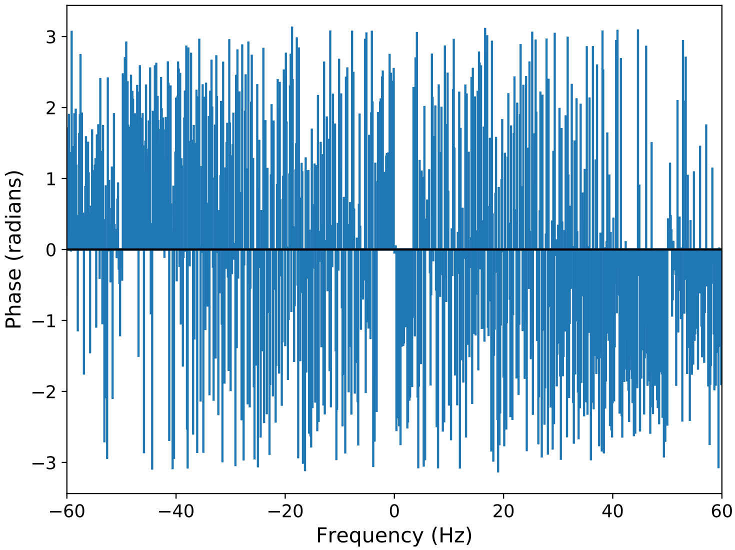
Electrocardiogram

$$\hat{x}[k] := \langle x, \phi_k \rangle = \int_0^T x(t) \exp\left(-\frac{i2\pi kt}{T}\right) \, \mathrm{d}t$$
$$= |\hat{x}[k]| \exp(\xi_k)$$

Electrocardiogram: Fourier coefficients (magnitude)



Electrocardiogram: Fourier coefficients (phase)



Real valued signals

Let $\hat{x}[k] = \alpha_k \exp(i\xi_k)$ be the Fourier series coefficients of a real-valued function $x \in \mathcal{L}_2[0, T]$

$$\hat{x}[-k] := \int_0^T x(t) \exp\left(\frac{i2\pi kt}{T}\right) \, \mathrm{d}t$$

$$= \overline{\int_0^T x(t) \exp\left(\frac{-i2\pi kt}{T}\right) \, \mathrm{d}t} \quad \text{because } x \text{ is real valued}$$

$$= \overline{\hat{x}[k]}$$

$$= \alpha_k \exp(-i\xi_k)$$

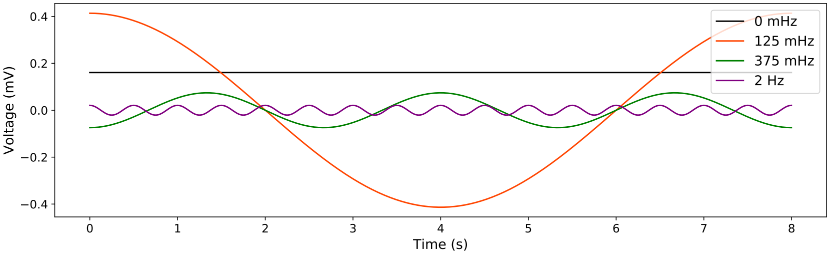
Real valued signals

Component corresponding to frequency $\frac{k}{T}$

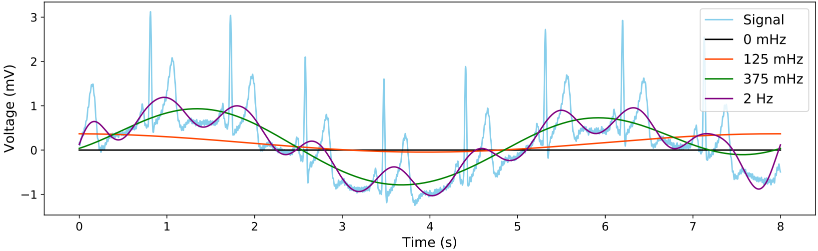
$$
\begin{aligned}
&\hat{x}[k]\phi_k + \hat{x}[-k]\phi_{-k} \\
&= \alpha_k \exp(i\xi_k) \exp\left(\frac{i2\pi kt}{T}\right) + \alpha_k \exp(-i\xi_k) \exp\left(-\frac{i2\pi kt}{T}\right) \\
&= \alpha_k \left(\exp\left(\frac{i2\pi kt}{T} + \xi_k\right) + \exp\left(-\frac{i2\pi kt}{T} - \xi_k\right)\right) \\
&= 2\alpha_k \cos\left(\frac{2\pi kt}{T} + \xi_k\right)
\end{aligned}
$$

$$
\mathcal{F}_{k_c}\{x\} := \frac{1}{T} \sum_{k=-k_c}^{k_c} \hat{x}[k]\phi_k = \alpha_0 + \frac{1}{T} \sum_{k=0}^{k_c} 2\alpha_k \cos\left(\frac{2\pi kt}{T} + \xi_k\right)
$$

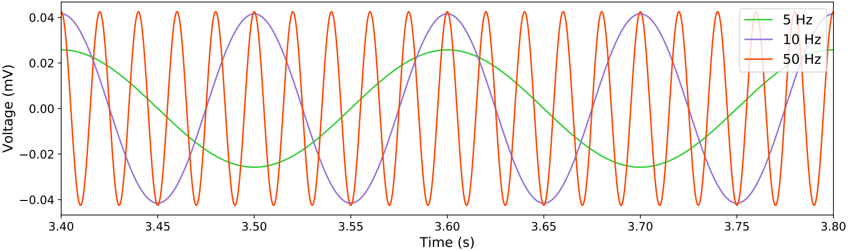
Fourier components



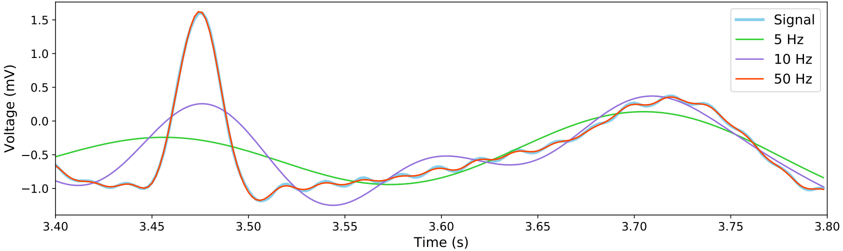
Fourier series



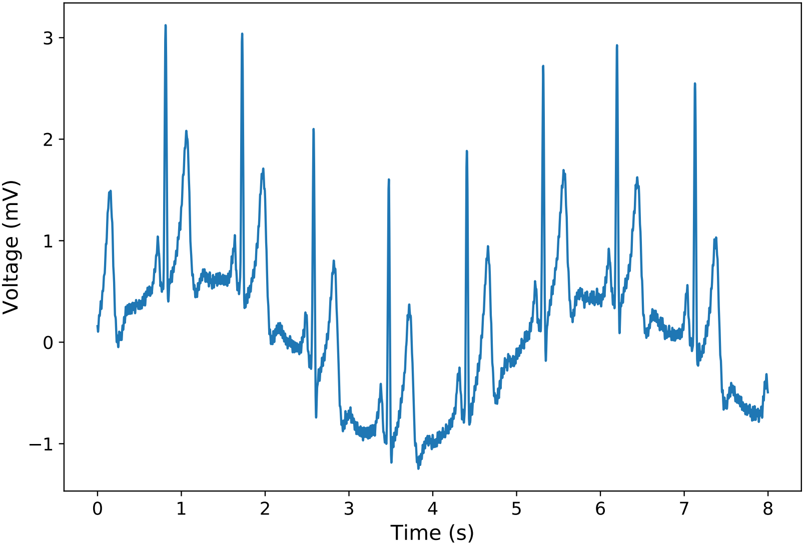
Fourier components



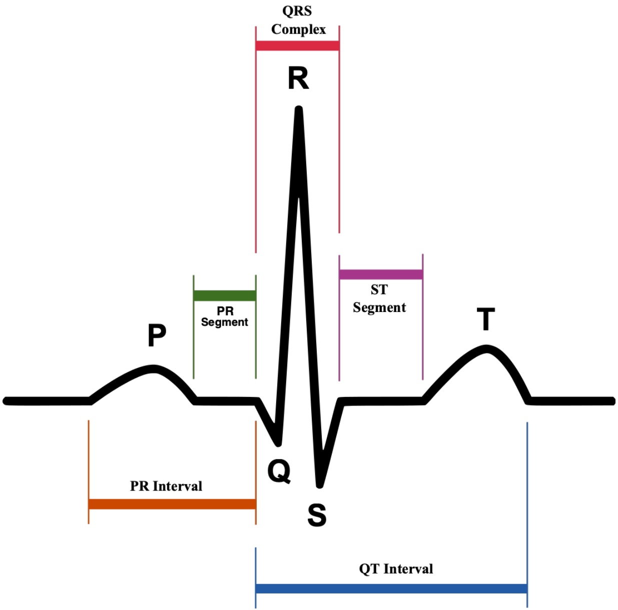
Fourier series



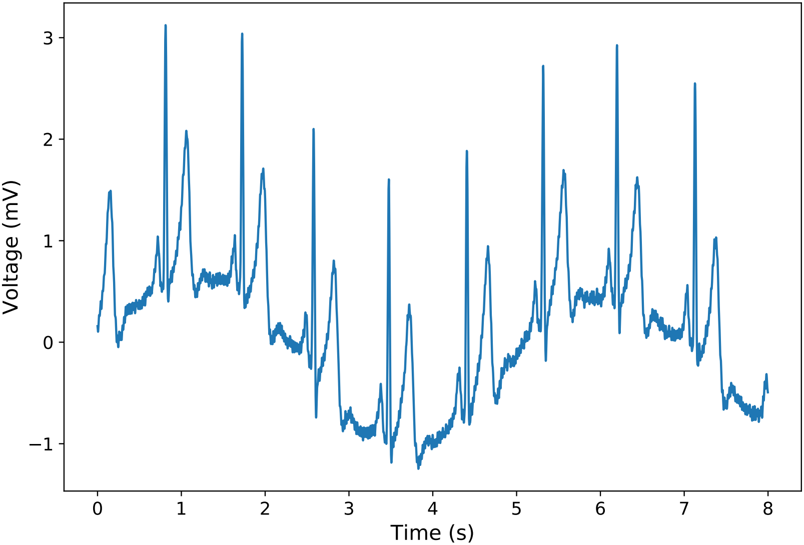
Electrocardiogram data



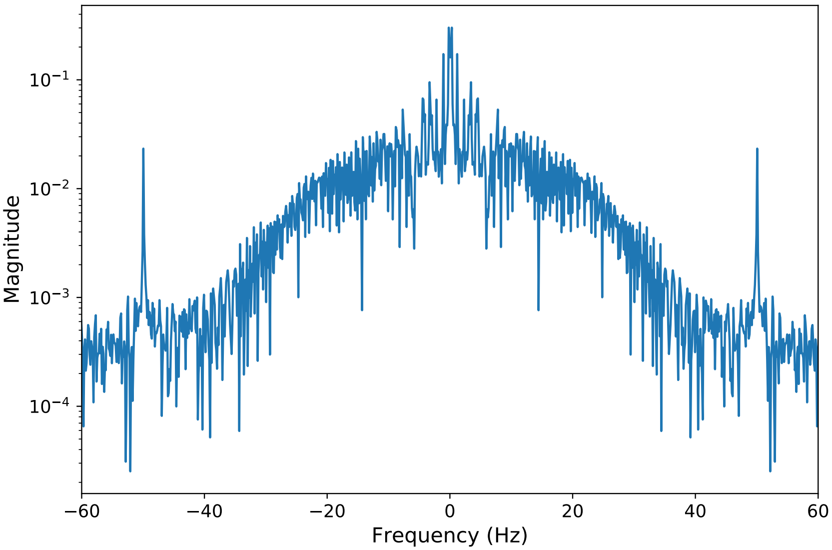
Electrocardiogram features



Problem: Baseline wandering



Electrocardiogram: Fourier coefficients (magnitude)



Filtering

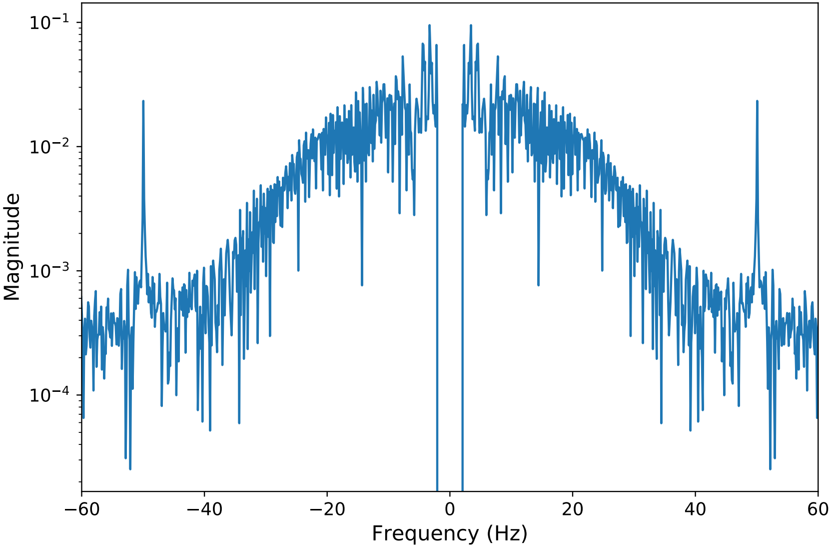
Idea: Process signal by removing (or attenuating) frequency components

High-pass filtering to correct baseline wandering

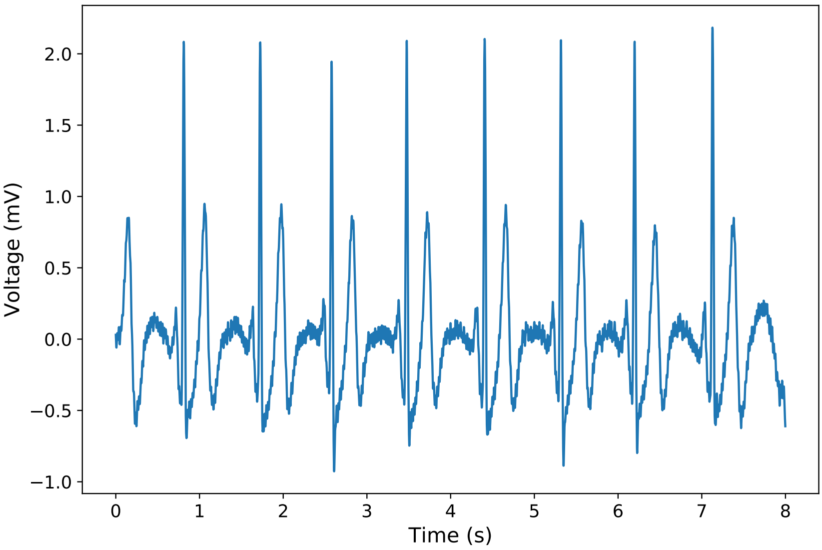
$$x \approx \frac{1}{T} \sum_{k=-k_{\text{max}}}^{-k_{\text{max}}} \hat{x}[k]\phi_k$$

$$x_{\text{high-pass}} := \frac{1}{T} \sum_{k=-k_{\text{max}}}^{-k_{\text{thresh}}} \hat{x}[k]\phi_k + \frac{1}{T} \sum_{k=k_{\text{thresh}}}^{k_{\text{max}}} \hat{x}[k]\phi_k$$

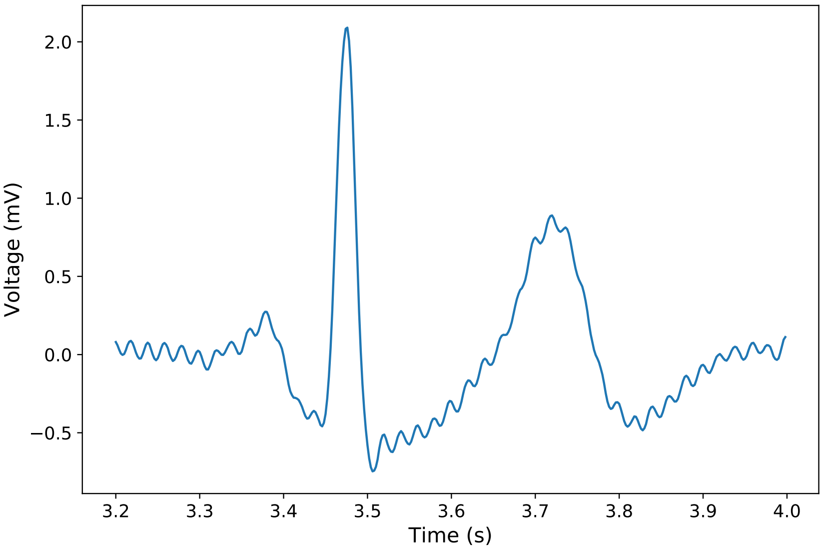
Electrocardiogram after high-pass filtering



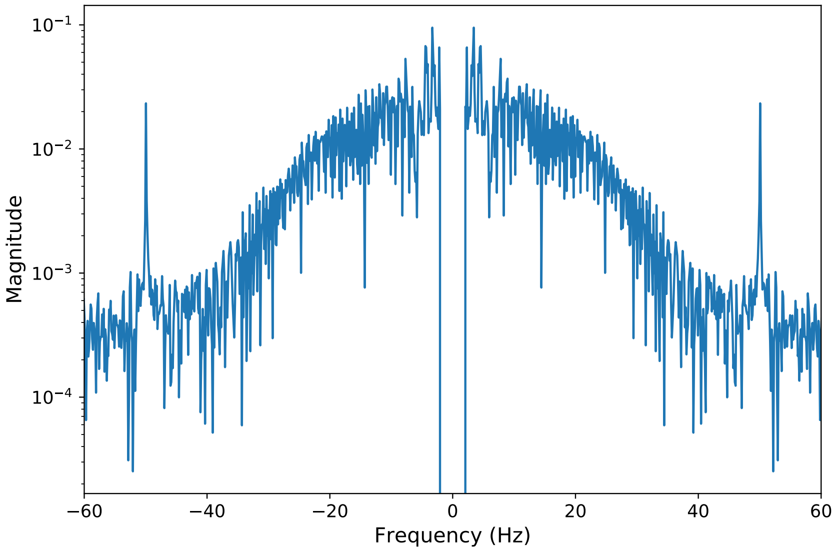
Electrocardiogram after high-pass filtering



Problem: Electric-grid interference



Fourier coefficients (magnitude)



Filtering

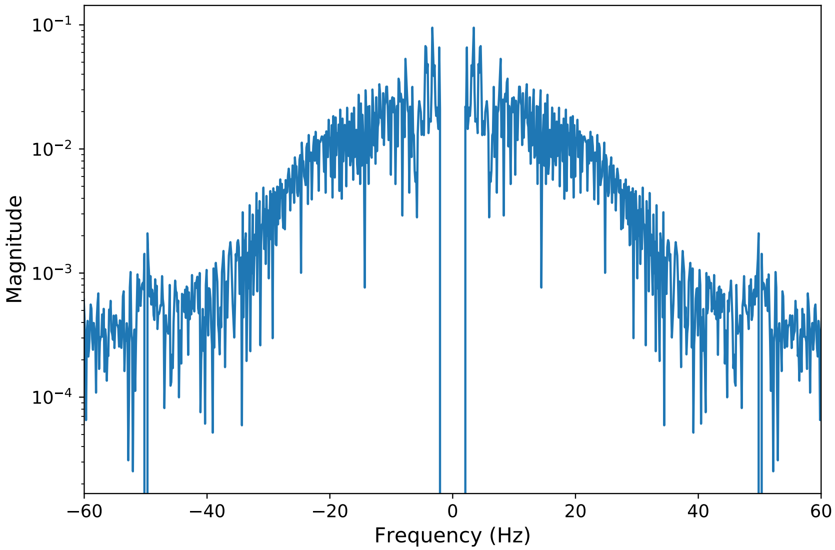
Idea: Can we remove the interference by filtering?

Band-stop filtering

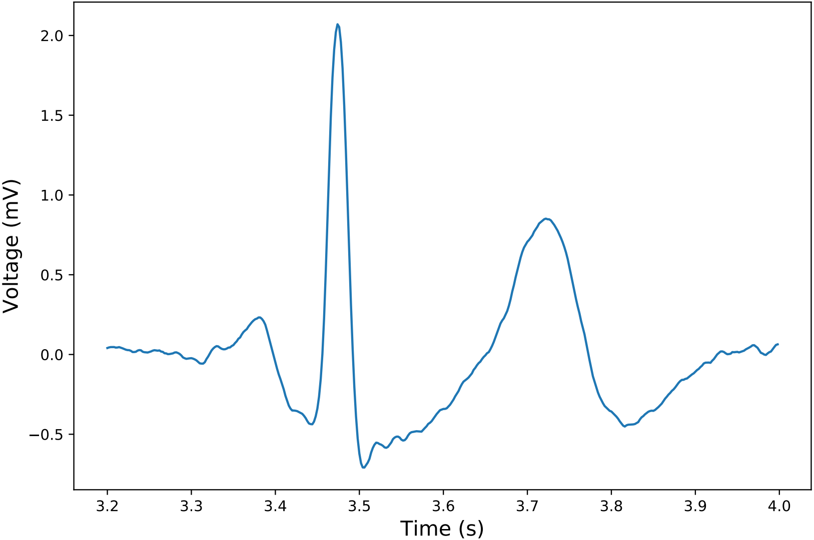
$$x_{\text{filtered}} := \frac{1}{T} \sum_{k=-k_{\text{max}}}^{-k_{\text{band-end}}} \hat{x}[k]\phi_k + \frac{1}{T} \sum_{k=-k_{\text{band-ini}}}^{-k_{\text{thresh}}} \hat{x}[k]\phi_k$$

$$+ \frac{1}{T} \sum_{k=k_{\text{thresh}}}^{k_{\text{band-ini}}} \hat{x}[k]\phi_k + \frac{1}{T} \sum_{k=k_{\text{band-end}}}^{k_{\text{max}}} \hat{x}[k]\phi_k$$

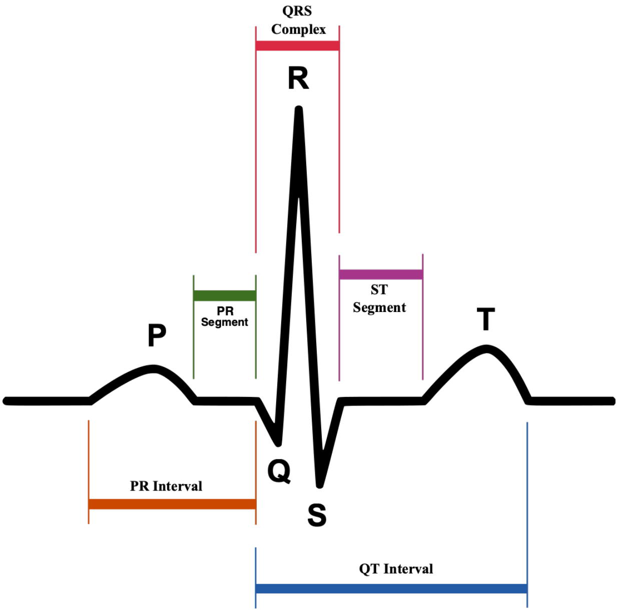
Filtered electrocardiogram



Filtered electrocardiogram



Electrocardiogram features



What have we learned

Complex exponentials with different frequencies form an orthogonal basis

This provides a representation of signals in terms of sinusoids

We can use this representation to filter different components of signals