Linear translation-invariant models and convolution

DS-GA 1013 / MATH-GA 2824 Mathematical Tools for Data Science

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Prerequisites

Linear regression

Fourier series

Discrete Fourier transform

Inverse problems

Goal: Estimate signal $y \in \mathbb{R}^N$ from noisy data $x \in \mathbb{R}^N$

Equivalent to a multivariate regression problem

Optimal estimator? Conditional mean of y given x

Are we done here? No, because of curse of dimensionality!

OK, let's use linear estimation then

Goal: Estimate signal $y \in \mathbb{R}^N$ from noisy data $x \in \mathbb{R}^N$

Idea: Learn linear function from $\mathbb{R}^N$ to $\mathbb{R}^N$ using least squares

For images $N \approx 10^4$, for audio $N \approx 10^6$ (sampling rate $\geq$ 40 kHz)

Number of parameters? 10^8 or 10^{12}!!!

We need further assumptions

Translation invariance: Shifting the noisy data, results in shifted estimate

Circular translation

We use circular translations that wrap around

We denote by $x^{\downarrow s}$ the sth circular translation of a vector $x \in \mathbb{C}^N$

For all $0 \leq j \leq N-1$,

$$x^{\downarrow s}[j] = x[(j-s) \bmod N]$$

Linear translation-invariant (LTI) function

A function $\mathcal{F}$ from $\mathbb{C}^N$ to $\mathbb{C}^N$ is linear if for any $x, y \in \mathbb{C}^N$ and any $\alpha \in \mathbb{C}$

$$\mathcal{F}(x + y) = \mathcal{F}(x) + \mathcal{F}(y),$$
$$\mathcal{F}(\alpha x) = \alpha \mathcal{F}(x),$$

and translation invariant if for any shift $0 \leq s \leq N - 1$

$$\mathcal{F}(x^{\downarrow s}) = \mathcal{F}(x)^{\downarrow s}$$

Parametrizing an LTI function

Let $\mathcal{F} : \mathbb{C}^N \to \mathbb{C}^N$ be linear and translation invariant

$$\begin{aligned}
y = \mathcal{F}_L(x) &= \mathcal{F}\left(\sum_{j=0}^{N-1} x[j]e_j\right) \\
&= \sum_{j=0}^{N-1} x[j]\mathcal{F}(e_j) \\
&= \sum_{j=0}^{N-1} x[j]\mathcal{F}\left(e_0^{\downarrow j}\right) \\
&= \sum_{j=0}^{N-1} x[j]\mathcal{F}(e_0)^{\downarrow j} \\
&= \sum_{j=0}^{N-1} x[j]h^{\downarrow j}
\end{aligned}$$

Impulse response

Standard basis vectors can be interpreted as *impulses*

LTI are characterized by their impulse response

$$h_{\mathcal{F}} := \mathcal{F}(e_0)$$

$$y[k] = \sum_{j=0}^{N-1} x[j] h^{\downarrow k}[j]$$

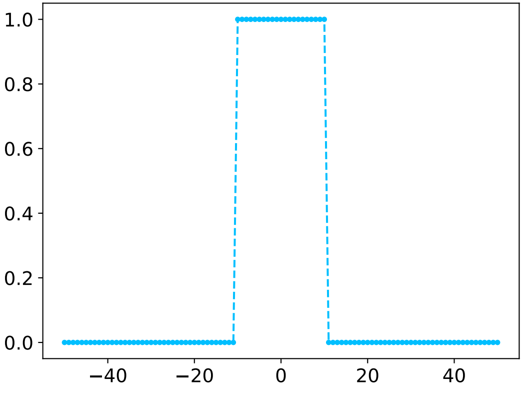
Number of parameters if signal/data is N dimensional? N!

Circular convolution

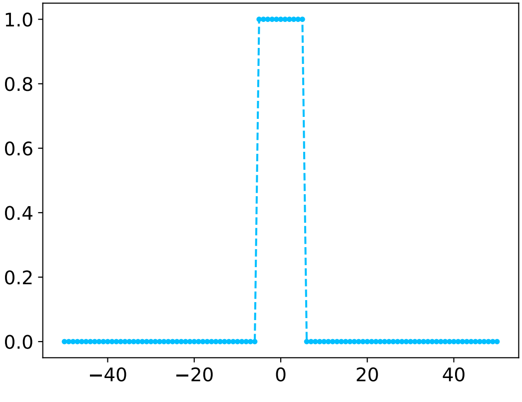
The circular convolution between two vectors $x, y \in \mathbb{C}^N$ is defined as

$$x * y\,[j] := \sum_{s=0}^{N-1} x\,[s]\,y^{\downarrow s}\,[j]\,, \quad 0 \leq j \leq N-1$$

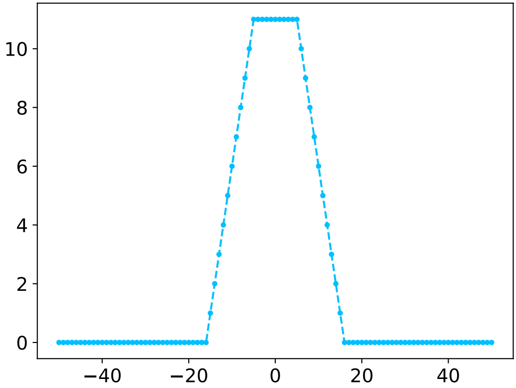
Convolution example: x



Convolution example: y



Convolution example: $x * y$

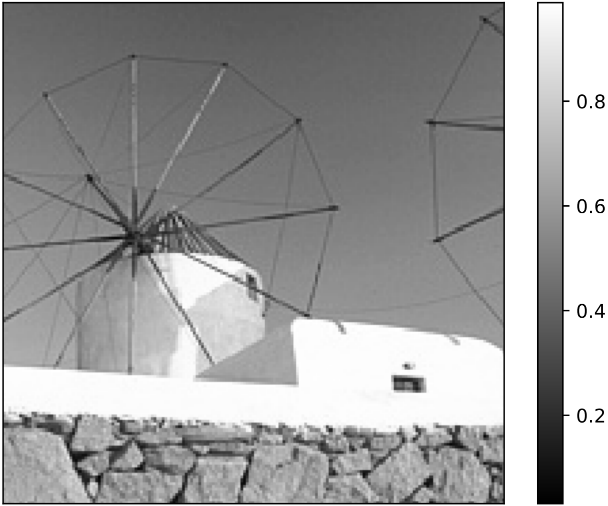


Circular convolution

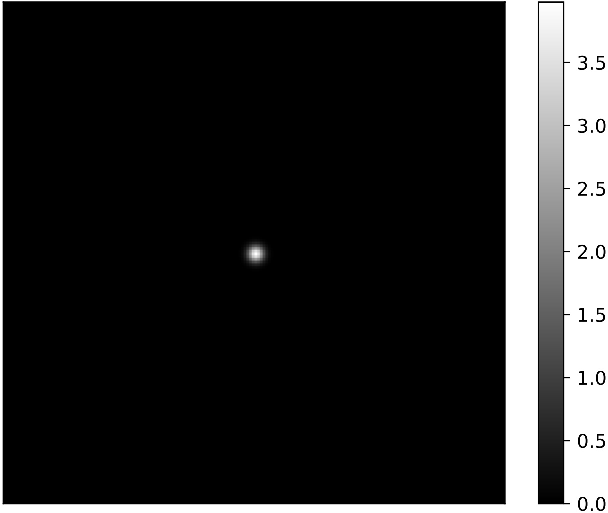
The 2D circular convolution between $X \in \mathbb{C}^{N \times N}$ and $Y \in \mathbb{C}^{N \times N}$ is

$$X * Y\,[j_1, j_2] := \sum_{s_1=0}^{N-1} \sum_{s_2=0}^{N-1} X\,[s_1, s_2]\, Y^{\downarrow(s_1,s_2)}\,[j_1, j_2]\,, \quad 0 \leq j_1, j_2 \leq N-1$$

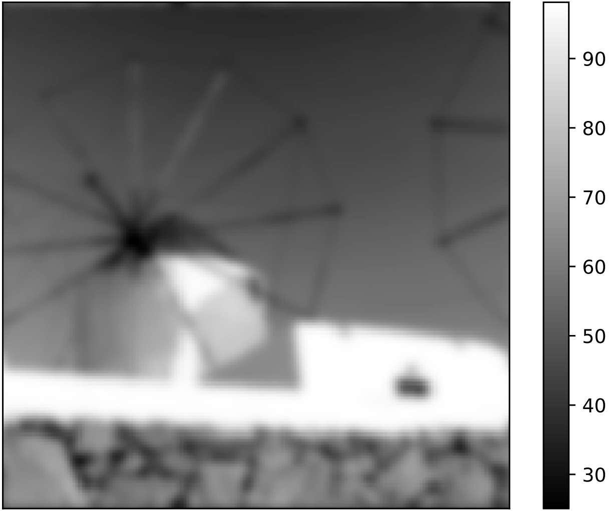
Convolution example: x



Convolution example: y



Convolution example: $x * y$



LTI functions as convolution with impulse response

For any LTI function $\mathcal{F} : \mathbb{C}^N \to \mathbb{C}^N$ and any $x \in \mathbb{C}^N$

$$\begin{aligned}\mathcal{F}(x) &= \sum_{j=0}^{N-1} x[j]\mathcal{F}(e_0)^{\downarrow j} \\ &= x * h_{\mathcal{F}}\end{aligned}$$

For any 2D LTI function $\mathcal{F} : \mathbb{C}^{N\times N} \to \mathbb{C}^{N\times N}$ and any $X \in \mathbb{C}^{N\times N}$

$$\mathcal{F}(X) = X * H_{\mathcal{F}}$$

Effect of translation on sinusoids

Translating a sinusoid modifies its phase

$$\psi_k^{\downarrow s}[l] = \exp\left(\frac{i2\pi k(l-s)}{N}\right)$$
$$= \exp\left(-\frac{i2\pi ks}{N}\right)\psi_k[l]$$

Effect of translation in Fourier domain

Let $x \in \mathbb{C}^N$ with DFT $\hat{x}$ and $y := x^{\downarrow s}$

$$
\begin{aligned}
\hat{y}[k] &:= \langle x^{\downarrow s}, \psi_k \rangle \\
&= \langle x, \psi_k^{\downarrow -s} \rangle \\
&= \left\langle x, \exp\left(\frac{i 2\pi k s}{N}\right) \psi_k \right\rangle \\
&= \exp\left(-\frac{i 2\pi k s}{N}\right) \langle x, \psi_k \rangle \\
&= \exp\left(-\frac{i 2\pi k s}{N}\right) \hat{x}[k]
\end{aligned}
$$

Convolution in the frequency domain

Let $y := x_1 * x_2$

$$
\begin{aligned}
\hat{y}[k] &:= \langle x_1 * x_2, \psi_k \rangle \\
&= \left\langle \sum_{s=0}^{N-1} x_1[s] \, x_2^{\downarrow s}, \psi_k \right\rangle \\
&= \left\langle \sum_{s=0}^{N-1} x_1[s] \, \frac{1}{N} \sum_{j=0}^{N-1} \exp\left(-\frac{i2\pi js}{N}\right) \hat{x}_2[j] \psi_j, \psi_k \right\rangle \\
&= \sum_{j=0}^{N-1} \hat{x}_2[j] \frac{1}{N} \langle \psi_j, \psi_k \rangle \sum_{s=0}^{N-1} x_1[s] \exp\left(-\frac{i2\pi js}{N}\right) \\
&= \sum_{j=0}^{N-1} \hat{x}_1[j] \hat{x}_2[j] \frac{1}{N} \langle \psi_j, \psi_k \rangle \\
&= \hat{x}_1[k] \hat{x}_2[k] \qquad \textit{Convolution in time is multiplication in frequency!}
\end{aligned}
$$

Convolution in time is multiplication in frequency

Let $y := x_1 * x_2$

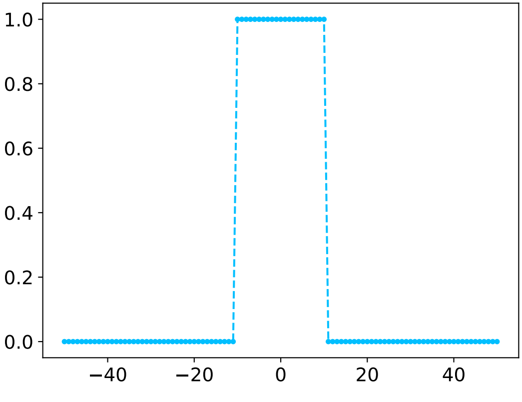
$$\hat{y}[k] = \hat{x}_1[k] \, \hat{x}_2[k], \quad 0 \leq k \leq N-1$$

Convolution in time is multiplication in frequency

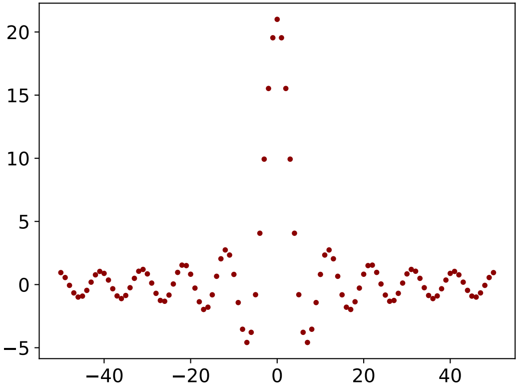
Let $Y := X_1 * X_2$ for $X_1, X_2 \in \mathbb{C}^{N \times N}$. Then

$$\widehat{Y}\,[k_1, k_2] = \widehat{X_1}\,[k_1, k_2]\,\widehat{X_2}\,[k_1, k_2]$$

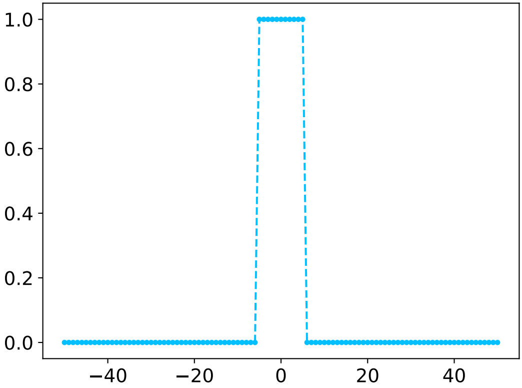
x



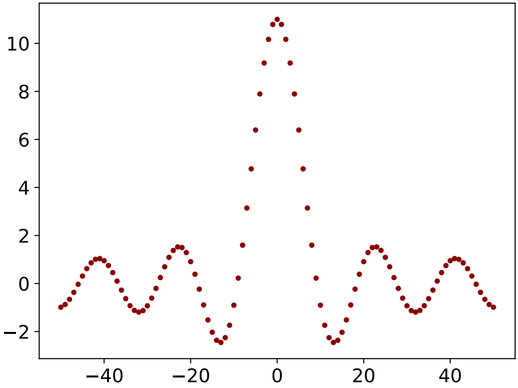
$\hat{x}$



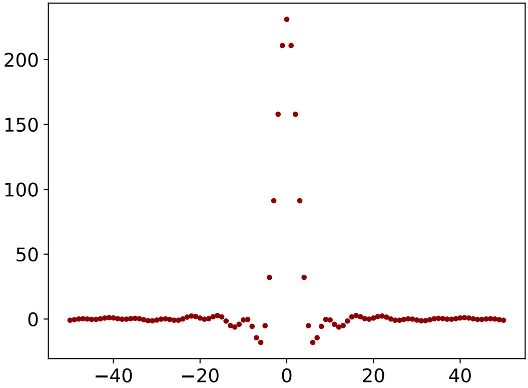
y



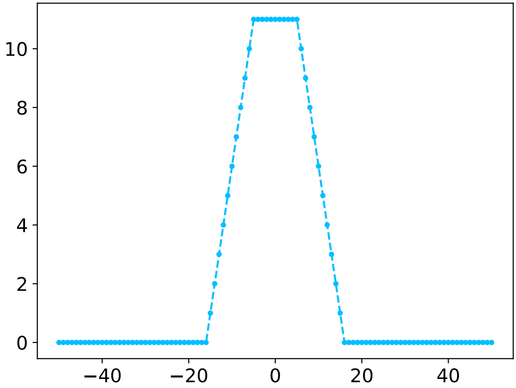
$\hat{y}$



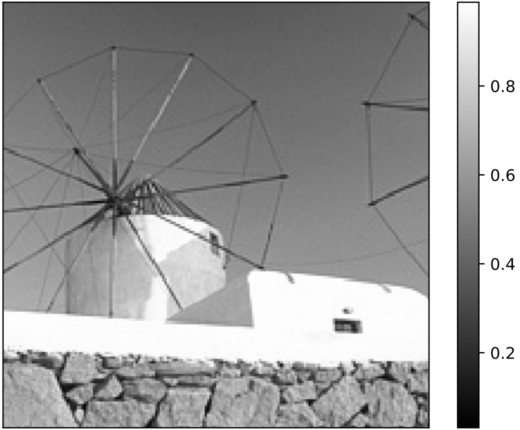
$\hat{x} \circ \hat{y}$



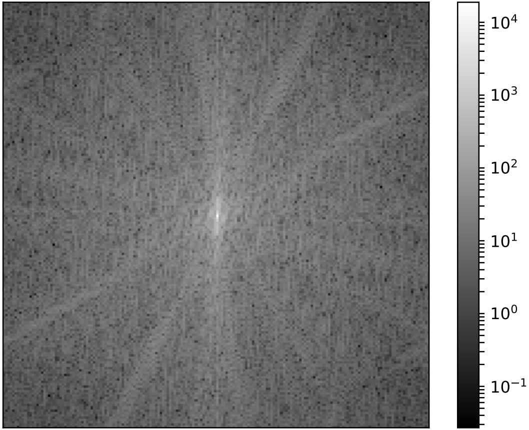
$x * y$

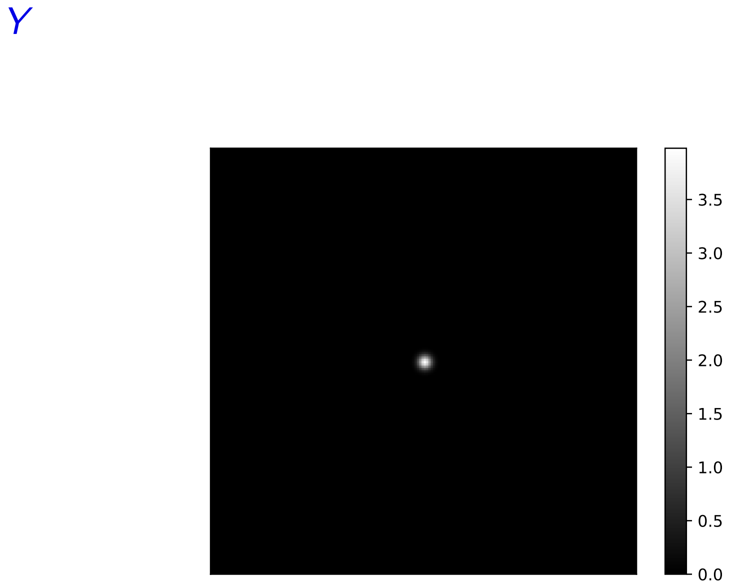


X

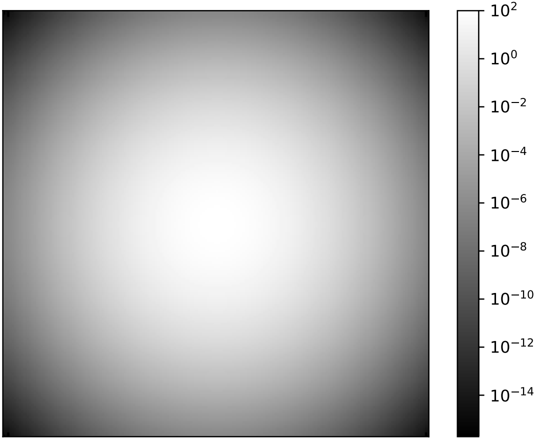


$\widehat{X}$

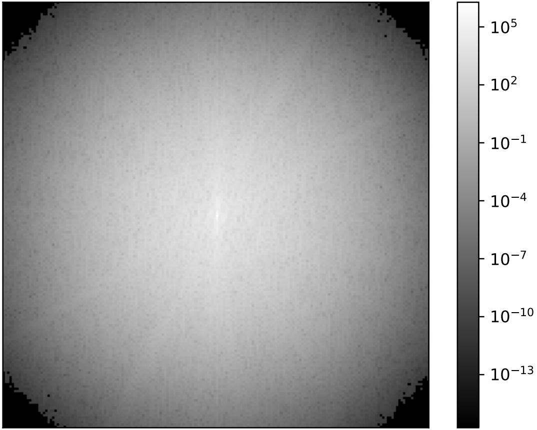



Y
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0.0

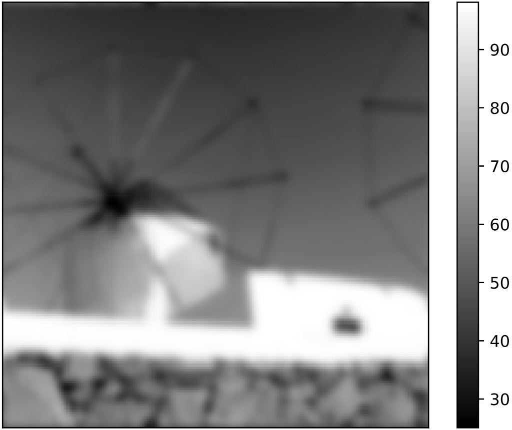
$\widehat{Y}$



$\widehat{X} \circ \widehat{Y}$



$X * Y$



Convolution in time is multiplication in frequency

LTI functions just scale Fourier coefficients!

DFT of impulse response is the transfer function of the function

For any LTI function $\mathcal{F}$ and any $x \in C^N$

$$\mathcal{F}(x) = \frac{1}{N} \sum_{k=0}^{N-1} \hat{h}_{\mathcal{F}}[k] \hat{x}[k] \psi_k.$$

For any 2D LTI function $\mathcal{F}$ and any $X \in C^{N \times N}$

$$\mathcal{F}(X) = \frac{1}{N^2} \sum_{k_1=0}^{N-1} \sum_{k_2=1}^{N} \hat{H}_{\mathcal{F}}[k_1, k_2] \widehat{X}[k_1, k_2] \Phi_{k_1, k_2}$$

What have we learned?

For high-dimensional signals fitting linear models is intractable

Assuming translation invariance yields efficient models

LTI models convolve the data with an impulse response

Equivalently, they scale each Fourier coefficient in the data individually