The Sampling Theorem

DS-GA 1013 / MATH-GA 2824 Mathematical Tools for Data Science

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Prerequisites

Calculus (complex numbers)

Linear algebra (orthogonality, basis, projections)

Fourier series
Sampling

Signals often model continuous objects

**Challenge:** How to measure them so that they can stored/processed

A common way is **sampling** their values at specific locations

**Crucial question:** Are we losing any information?
Sampling a complex sinusoid

We sample a complex sinusoid \( \phi_k(t) := \exp\left(\frac{i2\pi kt}{T}\right) \) in \([0, T)\) at \(N\) equidistant locations

\[
0, \frac{T}{N}, \frac{2T}{N}, \ldots, \frac{(N-1)T}{N}
\]
Can we distinguish it from complex sinusoids with other frequencies?
Sampling a complex sinusoid

\[ \phi_k \left( \frac{jT}{N} \right) = \exp \left( \frac{i2\pi kjT}{TN} \right) \]

\[ = \exp \left( \frac{i2\pi kj}{N} \right) \]

\[ = \exp \left( \frac{i2\pi kj}{N} + i2\pi pj \right) \]

\[ = \exp \left( \frac{i2\pi (k + pN)j}{N} \right) \]

\[ = \phi_{k+pN} \left( \frac{jT}{N} \right) \]

for any integer \( p \)
These frequencies yield the same samples:

\[ \ldots, \frac{k-2N}{T}, \frac{k-N}{T}, \frac{k}{T}, \frac{k+N}{T}, \frac{k+2N}{T}, \ldots \]

Can we at least distinguish between 0, \( \frac{1}{T} \), \( \frac{2}{T} \), \ldots, \( \frac{N-1}{T} \)?
Discrete complex sinusoids

Recall \( \phi_k \left( \frac{jT}{N} \right) = \exp \left( \frac{i2\pi k j T}{T N} \right) = \exp \left( \frac{i2\pi k j}{N} \right) \)

The discrete complex sinusoid \( \psi_k \in \mathbb{C}^N \) with frequency \( k \) is

\[
\psi_k [j] := \exp \left( \frac{i2\pi k j}{N} \right), \quad 0 \leq j, k \leq N - 1
\]
ψ_2 \ (N=10)
$\psi_3 \ (N=10)$
Inner product between discrete sinusoids

$$\langle \psi_k, \psi_l \rangle = \sum_{j=0}^{N-1} \psi_k[j] \overline{\psi_l[j]}$$

$$= \sum_{j=0}^{N-1} \exp \left( \frac{i2\pi(k-l)j}{N} \right) \left( = N \text{ if } k = l \text{ so } \|\psi_k\|_2 = \sqrt{N} \right)$$

$$= 1 - \exp \left( \frac{i2\pi(k-l)N}{N} \right)$$

$$= \frac{1 - \exp \left( \frac{i2\pi(k-l)N}{N} \right)}{1 - \exp \left( \frac{i2\pi(k-l)N}{N} \right)}$$

$$= 0 \quad \text{if } k \neq l$$

The discrete complex sinusoids form an orthogonal basis of $\mathbb{C}^N$
Bandlimited signals

A bandlimited signal cut-off frequency $k_c/T$ is equal to its Fourier series of order $k_c$

$$x(t) = \frac{1}{T} \sum_{k=-k_c}^{k_c} \hat{x}[k] \exp\left(\frac{i2\pi kt}{T}\right)$$

Bandlimited signals have a finite representation ($2k_c + 1$ coefficients)
Sampling a bandlimited signal on a uniform grid

Bandlimited signal $x$ measured at $N$ equispaced points in interval $T$

Samples: $x\left(\frac{0}{N}\right)$, $x\left(\frac{T}{N}\right)$, $x\left(\frac{2T}{N}\right)$, $\ldots$, $x\left(\frac{(N-1)T}{N}\right)$
Sampling a bandlimited signal on a uniform grid

Using Fourier series

\[ x \left( \frac{jT}{N} \right) = \frac{1}{T} \sum_{k=-k_c}^{k_c} \hat{x}_k \exp \left( \frac{i2\pi kjT}{NT} \right) \]

\[ = \frac{1}{T} \sum_{k=-k_c}^{k_c} \hat{x}_k \exp \left( \frac{i2\pi kj}{N} \right) \]

\[ = \frac{1}{T} \sum_{k=-k_c}^{k_c} \hat{x}_k \psi_k [j] \]

Vector of samples equals

\[ x[N] = \frac{1}{T} \sum_{k=-k_c}^{k_c} \hat{x}_k \psi_k \]
Sampling a bandlimited signal on a uniform grid

\[ x[N] = \frac{1}{T} \sum_{k=\pm k_c}^{k_c} \hat{x}_k \psi_k \]

We can recover the coefficients \( \hat{x}_k \) under 2 conditions:

1. There are more equations than unknowns \( N \geq 2k_c + 1 \)

2. \( \psi_{-k_c}, \psi_{-k_c+1}, \ldots, \psi_{k_c-1}, \psi_{k_c} \) are linearly independent
Sampling a bandlimited signal on a uniform grid

Let $N = 2k_c + 1$

In that case

$$\psi_{j-k_c} = \psi_{j-k_c+N} = \psi_{k_c+1+j}$$

In that case $\psi_{-k_c}, \psi_{-k_c+1}, \ldots, \psi_{-1}$ are $\psi_{k_c+1}, \psi_{k_c+2}, \ldots, \psi_{N-1}$

so $\psi_{-k_c}, \psi_{-k_c+1}, \ldots, \psi_{k_c-1}, \psi_{k_c}$ are $\psi_{0}, \psi_{1}, \ldots, \psi_{N-2}, \psi_{N-1}$

They are all orthogonal!
Sampling a bandlimited signal on a uniform grid

How do we recover the Fourier coefficients assuming $N = 2k_c + 1$?

$$x[N] = \frac{1}{T} \sum_{k=-k_c}^{k_c} \hat{x}_k \psi_k$$

$$\frac{T}{N} \left< x[N], \psi_k \right> = \frac{T}{N} \left< \frac{1}{T} \sum_{m=-k_c}^{k_c} \hat{x}_m \psi_m, \psi_k \right>$$

$$= \sum_{m=-k_c}^{k_c} \hat{x}_m \left< \frac{1}{\sqrt{N}} \psi_m, \frac{1}{\sqrt{N}} \psi_k \right>$$

$$= \hat{x}_k$$
Nyquist-Shannon-Kotelnikov sampling theorem

Any bandlimited signal \( x \in \mathcal{L}_2[0, T) \), where \( T > 0 \), with cut-off frequency \( k_c/T \) can be recovered exactly from \( N \) uniformly spaced samples \( x(0), x(T/N), \ldots, x(T - T/N) \) as long as

\[
N \geq 2k_c + 1,
\]

where \( 2k_c + 1 \) is known as the Nyquist rate

The Fourier series coefficients \( \hat{x} \) are recovered by computing

\[
\hat{x}_k = \frac{T}{N} \langle x[N], \psi_k \rangle
\]
Audio

Range of frequencies that human beings can hear is from 20 Hz to 20 kHz

At what frequency should we sample (at least)?

Typical rates used in practice: 44.1 kHz (CD), 48 kHz, 88.2 kHz, 96 kHz
Sampling a real sinusoid

Consider a real sinusoid with frequency equal to 4 Hz

\[ x(t) := \cos(8\pi t) = 0.5 \exp(-i2\pi 4t) + 0.5 \exp(i2\pi 4t) \]

measured over one second, i.e. \( T = 1 \) s

What is the cut-off frequency \( \frac{k_c}{T} \)? 4 Hz

Number of required samples \( N \)? \( 2k_c + 1 = 9 \)
$N = 10$

\[
x[10] = \begin{bmatrix}
  x(0) \\
x(\frac{1}{10}) \\
  \vdots \\
x(\frac{9}{10})
\end{bmatrix}
\]

\[
= 0.5 \begin{bmatrix}
  \exp(-i 2\pi 4 \cdot 0) \\
  \exp(-i 2\pi 4 \cdot \frac{1}{10}) \\
  \vdots \\
  \exp(-i 2\pi 4 \cdot \frac{9}{10})
\end{bmatrix} + 0.5 \begin{bmatrix}
  \exp(i 2\pi 4 \cdot 0) \\
  \exp(i 2\pi 4 \cdot \frac{1}{10}) \\
  \vdots \\
  \exp(i 2\pi 4 \cdot \frac{9}{10})
\end{bmatrix}
\]

\[
= 0.5 \psi_{-4} + 0.5 \psi_4
\]
Recovery

\( N = 10 \), so \( \psi_{−4}, \psi_{−3}, \ldots, \psi_{3}, \psi_{4} \) are orthogonal

\[
\hat{x}_{\text{rec}}[k] = \frac{T}{N} \langle x[N], \psi_k \rangle
\]

\[
\hat{x}_{\text{rec}}[-4] = \frac{1}{9} \langle 0.5\psi_{−4} + 0.5\psi_{4}, \psi_{−4} \rangle = 0.5
\]

\[
\hat{x}_{\text{rec}}[4] = \frac{1}{9} \langle 0.5\psi_{−4} + 0.5\psi_{4}, \psi_{4} \rangle = 0.5
\]

\[
\hat{x}_{\text{rec}}[k] = \frac{1}{9} \langle 0.5\psi_{−4} + 0.5\psi_{4}, \psi_k \rangle = 0 \quad k \in \{-3, -2, -1, 0, 1, 2, 3\}
\]
Recovered Fourier coefficients ($N = 10$)

![Plot of recovered Fourier coefficients with frequency on the x-axis and magnitude on the y-axis. The plot includes points for the signal and reconstruction.]
Recovered signal ($N = 10$)
What if $N = 5$ and we assume (mistakenly) $k_c = 2$?

Remember that $\psi_{k+pN} = \psi_{k+5p} = \psi_k$ for any $p$, so

$$x[5] = \begin{bmatrix} x(0) \\ x\left(\frac{1}{5}\right) \\ \vdots \\ x\left(\frac{4}{5}\right) \end{bmatrix} = 0.5\psi_{-4} + 0.5\psi_4 = 0.5\psi_1 + 0.5\psi_{-1}$$
What if \( N = 5 \) and we assume (mistakenly) \( k_c = 2 \)?

\[
\hat{x}^{\text{rec}}[k] = \frac{T}{N} \langle x[N], \psi_k \rangle
\]

\[
\begin{align*}
\hat{x}^{\text{rec}}[-2] &= \frac{1}{5} \langle 0.5 \psi_1 + 0.5 \psi_{-1}, \psi_{-2} \rangle = 0 \\
\hat{x}^{\text{rec}}[-1] &= \frac{1}{5} \langle 0.5 \psi_1 + 0.5 \psi_{-1}, \psi_{-1} \rangle = 0.5 \\
\hat{x}^{\text{rec}}[0] &= \frac{1}{5} \langle 0.5 \psi_1 + 0.5 \psi_{-1}, \psi_0 \rangle = 0 \\
\hat{x}^{\text{rec}}[1] &= \frac{1}{5} \langle 0.5 \psi_1 + 0.5 \psi_{-1}, \psi_1 \rangle = 0.5 \\
\hat{x}^{\text{rec}}[2] &= \frac{1}{5} \langle 0.5 \psi_1 + 0.5 \psi_{-1}, \psi_2 \rangle = 0
\end{align*}
\]
Recovered Fourier coefficients \( (N = 5) \)
Recovered signal \((N = 5)\)

\[
\chi^{\text{rec}}(t) = \hat{\chi}^{\text{rec}}[-1] \exp(-2\pi t) + \hat{\chi}^{\text{rec}}[1] \exp(2\pi t)
\]

\[
= \cos(2\pi t) \neq \cos(8\pi t) \quad \text{Aliasing!}
\]
Recovered signal \((N = 5)\)
Aliasing

Let $x$ be a signal with cut-off frequency $k_{\text{true}}/T$

We measure $x[N]$, $N$ samples of $x$ at $0$, $T/N$, $2T/N$, ... $T - T/N$

What happens if we recover the signal assuming it is bandlimited with cut-off freq $k_{\text{samp}}/T$, $N = 2k_{\text{samp}} + 1$, but actually $k_{\text{true}} > k_{\text{samp}}$?

$$\hat{x}^{\text{rec}}[k] := \frac{T}{N} \langle \psi_k, x[N] \rangle$$

$$= \frac{T}{N} \left\langle \frac{1}{T} \sum_{m=-k_{\text{true}}}^{k_{\text{true}}} \hat{x}[m] \psi_m, \psi_k \right\rangle$$

$$= \frac{1}{N} \sum_{m=-k_{\text{true}}}^{k_{\text{true}}} \hat{x}[m] \langle \psi_m, \psi_k \rangle$$

$$= \sum_{\{(m-k) \mod N = 0\}} \hat{x}[m]$$
Electrocardiogram: Fourier coefficients (magnitude)
Sampling an electrocardiogram

Signal is approximately bandlimited at $\frac{k_c}{T} = 50 \text{ Hz}$

$T = 8 \text{ s}$, so $k_c = 50T = 400$

To avoid aliasing $N \geq 801$
Recovered Fourier coefficients ($N=1,000$)
We mistakenly assume that the signal is approximately bandlimited at around 40 Hz so $k_c = 312$ and $N = 625$

\[
\hat{x}_{\text{rec}}[k] = \sum_{\{(m-k) \mod 625=0\}} \hat{x}[m]
\]

Component at $m = \pm 400$ (50 Hz) shows up at $\pm 225$ (28.1 Hz)
Recovered Fourier coefficients ($N = 625$)
Recovered signal \((N = 625)\)
What have we learned

Definition of orthogonal basis of discrete complex sinusoids

How to recover bandlimited signals from a finite number of samples

That insufficient sampling leads to aliasing