The Sampling Theorem

DS-GA 1013 / MATH-GA 2824 Mathematical Tools for Data Science

Carlos Fernandez-Granda

Prerequisites

Calculus (complex numbers)

Linear algebra (orthogonality, basis, projections)

Fourier series

Sampling

Signals often model continuous objects

Challenge: How to measure them so that they can stored/processed

A common way is sampling their values at specific locations

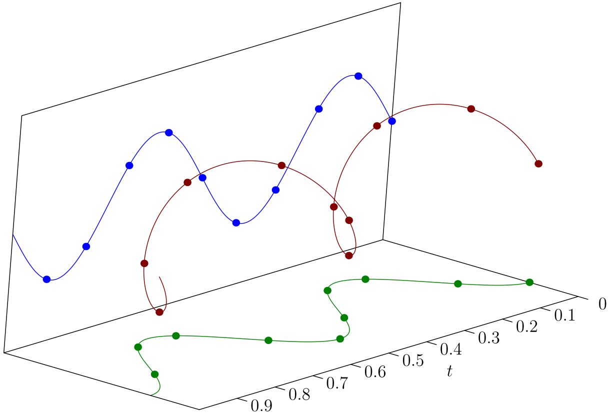
Crucial question: Are we losing any information?

Sampling a complex sinusoid

We sample a complex sinusoid $\phi_k(t) := \exp\left(\frac{i2\pi kt}{T}\right)$ in $[0, T)$ at N equidistant locations

$$0, \frac{T}{N}, \frac{2T}{N}, \ldots, \frac{(N-1)T}{N}$$

Can we distinguish it from complex sinusoids with other frequencies?



Sampling a complex sinusoid

$$
\begin{aligned}
\phi_k\left(\frac{jT}{N}\right) &= \exp\left(\frac{i2\pi kjT}{TN}\right) \\
&= \exp\left(\frac{i2\pi kj}{N}\right) \\
&= \exp\left(\frac{i2\pi kj}{N} + i2\pi pj\right) \qquad \text{for any integer } p \\
&= \exp\left(\frac{i2\pi(k+pN)j}{N}\right) \\
&= \phi_{k+pN}\left(\frac{jT}{N}\right)
\end{aligned}
$$

Sampling a complex sinusoid

These frequencies yield the same samples:

$$\ldots, \frac{k-2N}{T}, \frac{k-N}{T}, \frac{k}{T}, \frac{k+N}{T}, \frac{k+2N}{T}, \ldots$$

Can we at least distinguish between 0, $\frac{1}{T}$, $\frac{2}{T}$, $\ldots$, $\frac{N-1}{T}$?

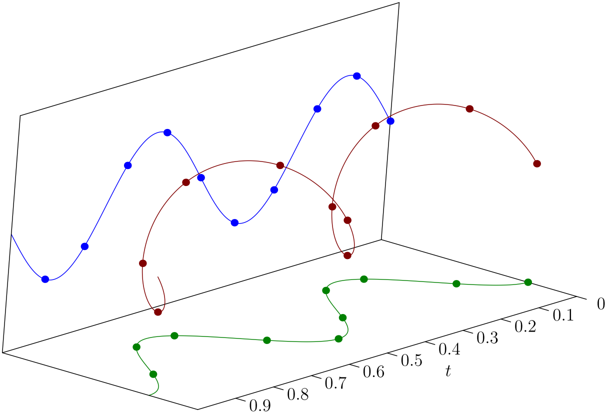
Discrete complex sinusoids

Recall $\phi_k\left(\frac{jT}{N}\right) = \exp\left(\frac{i2\pi kjT}{TN}\right) = \exp\left(\frac{i2\pi kj}{N}\right)$

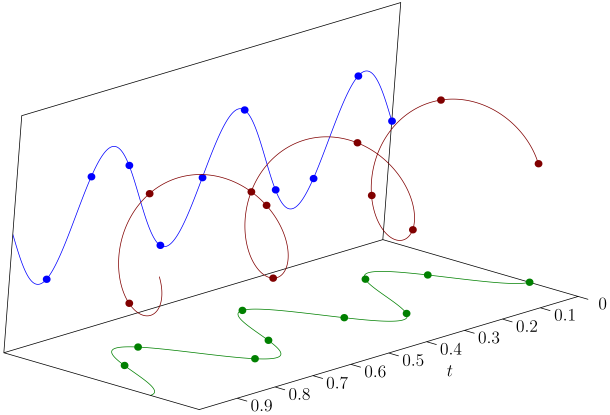
The discrete complex sinusoid $\psi_k \in \mathbb{C}^N$ with frequency k is

$$\psi_k[j] := \exp\left(\frac{i2\pi kj}{N}\right), \qquad 0 \leq j, k \leq N-1$$

ψ_2 (N=10)



ψ_3 (N=10)



Inner product between discrete sinusoids

$$\begin{aligned}
\langle \psi_k, \psi_l \rangle &= \sum_{j=0}^{N-1} \psi_k[j] \overline{\psi_l[j]} \\
&= \sum_{j=0}^{N-1} \exp\left(\frac{i 2\pi (k-l) j}{N} \right) \quad \left(= N \quad \text{if } k = l \text{ so } ||\psi_k||_2 = \sqrt{N} \right) \\
&= \frac{1 - \exp\left(\frac{i2\pi(k-l)N}{N} \right)}{1 - \exp\left(\frac{i2\pi(k-l)}{N} \right)} \\
&= 0 \qquad \text{if } k \neq l
\end{aligned}$$

The discrete complex sinusoids form an orthogonal basis of $\mathbb{C}^N$

Bandlimited signals

A bandlimited signal cut-off frequency k_c/T is equal to its Fourier series of order k_c

$$x(t) = \frac{1}{T} \sum_{k=-k_c}^{k_c} \hat{x}[k] \exp\left(\frac{i2\pi kt}{T}\right)$$

Bandlimited signals have a finite representation ($2k_c + 1$ coefficients)

Sampling a bandlimited signal on a uniform grid

Bandlimited signal x measured at N equispaced points in interval T

Samples: $x\left(\frac{0}{N}\right)$, $x\left(\frac{T}{N}\right)$, $x\left(\frac{2T}{N}\right)$, ..., $x\left(\frac{(N-1)T}{N}\right)$

Sampling a bandlimited signal on a uniform grid

Using Fourier series

$$x\left(\frac{jT}{N}\right) = \frac{1}{T}\sum_{k=-k_c}^{k_c} \hat{x}_k \exp\left(\frac{i2\pi k jT}{NT}\right)$$

$$= \frac{1}{T}\sum_{k=-k_c}^{k_c} \hat{x}_k \exp\left(\frac{i2\pi kj}{N}\right)$$

$$= \frac{1}{T}\sum_{k=-k_c}^{k_c} \hat{x}_k \psi_k[j]$$

Vector of samples equals

$$x_{[N]} = \frac{1}{T}\sum_{k=-k_c}^{k_c} \hat{x}_k \psi_k$$

Sampling a bandlimited signal on a uniform grid

$$x_{[N]} = \frac{1}{T} \sum_{k=-k_c}^{k_c} \hat{x}_k \psi_k$$

We can recover the coefficients $\hat{x}_k$? under 2 conditions:

1. There are more equations than unknowns $N \geq 2k_c + 1$
2. $\psi_{-k_c}, \psi_{-k_c+1}, \ldots, \psi_{k_c-1}, \psi_{k_c}$ are linearly independent

Sampling a bandlimited signal on a uniform grid

Let $N = 2k_c + 1$

In that case

$$\begin{aligned}\psi_{j-k_c} &= \psi_{j-k_c+N} \\ &= \psi_{k_c+1+j}\end{aligned}$$

In that case $\psi_{-k_c}, \psi_{-k_c+1}, \ldots, \psi_{-1}$ are $\psi_{k_c+1}, \psi_{k_c+2}, \ldots, \psi_{N-1}$

so $\psi_{-k_c}, \psi_{-k_c+1}, \ldots, \psi_{k_c-1}, \psi_{k_c}$ are $\psi_0, \psi_1, \ldots, \psi_{N-2}, \psi_{N-1}$

They are all orthogonal!

Sampling a bandlimited signal on a uniform grid

How do we recover the Fourier coefficients assuming $N = 2k_c + 1$?

$$x_{[N]} = \frac{1}{T} \sum_{k=-k_c}^{k_c} \hat{x}_k \psi_k$$

$$\begin{aligned}
\frac{T}{N} \left\langle x_{[N]}, \psi_k \right\rangle &= \frac{T}{N} \left\langle \frac{1}{T} \sum_{m=-k_c}^{k_c} \hat{x}_m \psi_m, \psi_k \right\rangle \\
&= \sum_{m=-k_c}^{k_c} \hat{x}_m \left\langle \frac{1}{\sqrt{N}} \psi_m, \frac{1}{\sqrt{N}} \psi_k \right\rangle \\
&= \hat{x}_k
\end{aligned}$$

Nyquist-Shannon-Kotelnikov sampling theorem

Any bandlimited signal $x \in \mathcal{L}_2[0, T)$, where $T > 0$, with cut-off frequency k_c/T can be recovered exactly from N uniformly spaced samples $x(0)$, $x(T/N)$, $\ldots$, $x(T - T/N)$ as long as

$$N \geq 2k_c + 1,$$

where $2k_c + 1$ is known as the Nyquist rate

The Fourier series coefficients $\hat{x}$ are recovered by computing

$$\hat{x}_k = \frac{T}{N} \left\langle x_{[N]}, \psi_k \right\rangle$$

Audio

Range of frequencies that human beings can hear is from 20 Hz to 20 kHz

At what frequency should we sample (at least)?

Typical rates used in practice: 44.1 kHz (CD), 48 kHz, 88.2 kHz, 96 kHz

Sampling a real sinusoid

Consider a real sinusoid with frequency equal to 4 Hz

$$\begin{aligned} x(t) &:= \cos(8\pi t) \\ &= 0.5 \exp(-i 2\pi 4 t) + 0.5 \exp(i 2\pi 4 t) \end{aligned}$$

measured over one second, i.e. $T = 1$ s

What is the cut-off frequency $\frac{k_c}{T}$? 4 Hz

Number of required samples N? $2k_c + 1 = 9$

$N = 10$

$$x_{[10]} = \begin{bmatrix} x(0) \\ x\left(\frac{1}{10}\right) \\ \cdots \\ x\left(\frac{9}{10}\right) \end{bmatrix}$$

$$= 0.5 \begin{bmatrix} \exp\left(-i2\pi 4 \cdot 0\right) \\ \exp\left(-i2\pi 4 \cdot \frac{1}{10}\right) \\ \cdots \\ \exp\left(-i2\pi 4 \cdot \frac{9}{10}\right) \end{bmatrix} + 0.5 \begin{bmatrix} \exp(i2\pi 4 \cdot 0) \\ \exp\left(i2\pi 4 \cdot \frac{1}{10}\right) \\ \cdots \\ \exp\left(i2\pi 4 \cdot \frac{9}{10}\right) \end{bmatrix}$$

$$= 0.5\psi_{-4} + 0.5\psi_{4}$$

Recovery

$N = 10$, so $\psi_{-4}, \psi_{-3}, \ldots, \psi_3, \psi_4$ are orthogonal

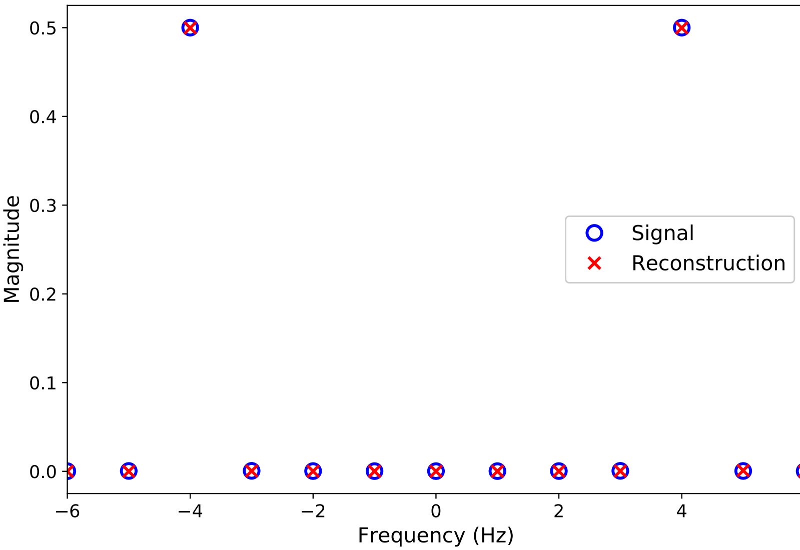
$$\hat{x}^{\text{rec}}[k] = \frac{T}{N} \left\langle x_{[N]}, \psi_k \right\rangle$$

$$\hat{x}^{\text{rec}}[-4] = \frac{1}{9} \langle 0.5\psi_{-4} + 0.5\psi_4, \psi_{-4} \rangle = 0.5$$

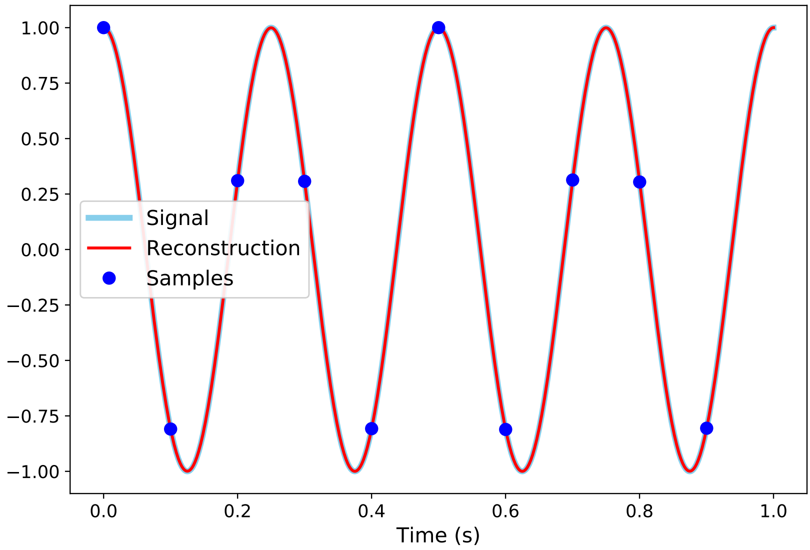
$$\hat{x}^{\text{rec}}[4] = \frac{1}{9} \langle 0.5\psi_{-4} + 0.5\psi_4, \psi_4 \rangle = 0.5$$

$$\hat{x}^{\text{rec}}[k] = \frac{1}{9} \langle 0.5\psi_{-4} + 0.5\psi_4, \psi_k \rangle = 0 \qquad k \in \{-3, -2, -1, 0, 1, 2, 3\}$$

Recovered Fourier coefficients ($N = 10$)



Recovered signal ($N = 10$)



What if $N = 5$ and we assume (mistakenly) $k_c = 2$?

Remember that $\psi_{k+pN} = \psi_{k+5p} = \psi_k$ for any p, so

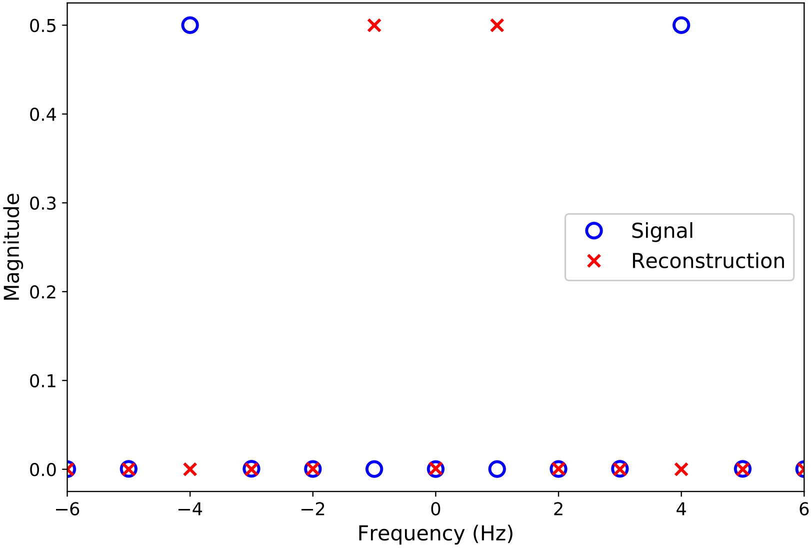
$$x_{[5]} = \begin{bmatrix} x(0) \\ x\left(\frac{1}{5}\right) \\ \cdots \\ x\left(\frac{4}{5}\right) \end{bmatrix} = 0.5\psi_{-4} + 0.5\psi_4 = 0.5\psi_1 + 0.5\psi_{-1}$$

What if $N = 5$ and we assume (mistakenly) $k_c = 2$?

$$\hat{x}^{\text{rec}}[k] = \frac{T}{N} \left\langle x_{[N]}, \psi_k \right\rangle$$

$$\hat{x}^{\text{rec}}[-2] = \frac{1}{5} \langle 0.5\psi_1 + 0.5\psi_{-1}, \psi_{-2} \rangle = 0$$
$$\hat{x}^{\text{rec}}[-1] = \frac{1}{5} \langle 0.5\psi_1 + 0.5\psi_{-1}, \psi_{-1} \rangle = 0.5$$
$$\hat{x}^{\text{rec}}[0] = \frac{1}{5} \langle 0.5\psi_1 + 0.5\psi_1, \psi_0 \rangle = 0$$
$$\hat{x}^{\text{rec}}[1] = \frac{1}{5} \langle 0.5\psi_1 + 0.5\psi_1, \psi_1 \rangle = 0.5$$
$$\hat{x}^{\text{rec}}[2] = \frac{1}{5} \langle 0.5\psi_1 + 0.5\psi_1, \psi_2 \rangle = 0$$

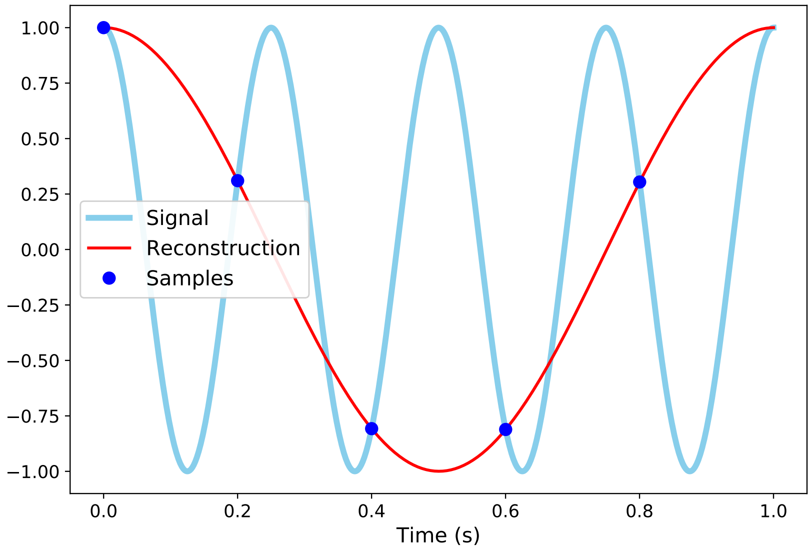
Recovered Fourier coefficients ($N = 5$)



Recovered signal ($N = 5$)

$$
\begin{aligned}
x^{\text{rec}}(t) &= \hat{x}^{\text{rec}}[-1] \exp(-2\pi t) + \hat{x}^{\text{rec}}[1] \exp(2\pi t) \\
&= \cos(2\pi t) \neq \cos(8\pi t) \qquad \text{Aliasing!}
\end{aligned}
$$

Recovered signal ($N = 5$)



Aliasing

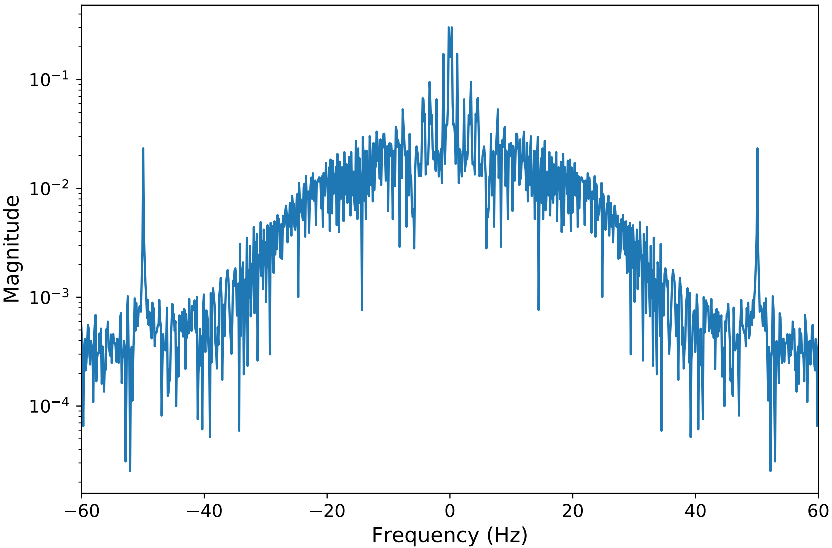
Let x be a signal with cut-off frequency k_{true}/T

We measure $x_{[N]}$, N samples of x at 0, T/N, $2T/N$, $\ldots T - T/N$

What happens if we recover the signal assuming it is bandlimited with cut-off freq k_{samp}/T, $N = 2k_{\text{samp}} + 1$, but actually $k_{\text{true}} > k_{\text{samp}}$?

$$\begin{aligned}
\hat{x}^{\text{rec}}[k] &:= \frac{T}{N} \langle \psi_k, x_{[N]} \rangle \\
&= \frac{T}{N} \left\langle \frac{1}{T} \sum_{m=-k_{\text{true}}}^{k_{\text{true}}} \hat{x}[m] \psi_m, \psi_k \right\rangle \\
&= \frac{1}{N} \sum_{m=-k_{\text{true}}}^{k_{\text{true}}} \hat{x}[m] \langle \psi_m, \psi_k \rangle \\
&= \sum_{\{(m-k) \bmod N = 0\}} \hat{x}[m]
\end{aligned}$$

Electrocardiogram: Fourier coefficients (magnitude)



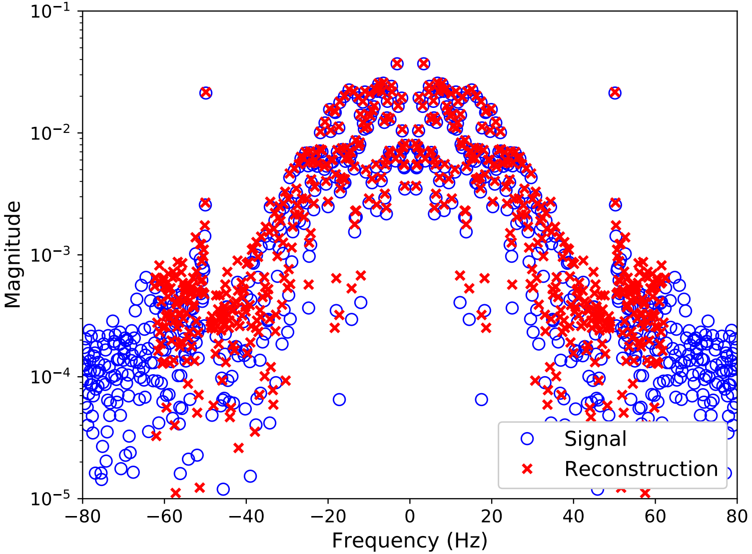
Sampling an electrocardiogram

Signal is approximately bandlimited at $\frac{k_c}{T} = 50$ Hz

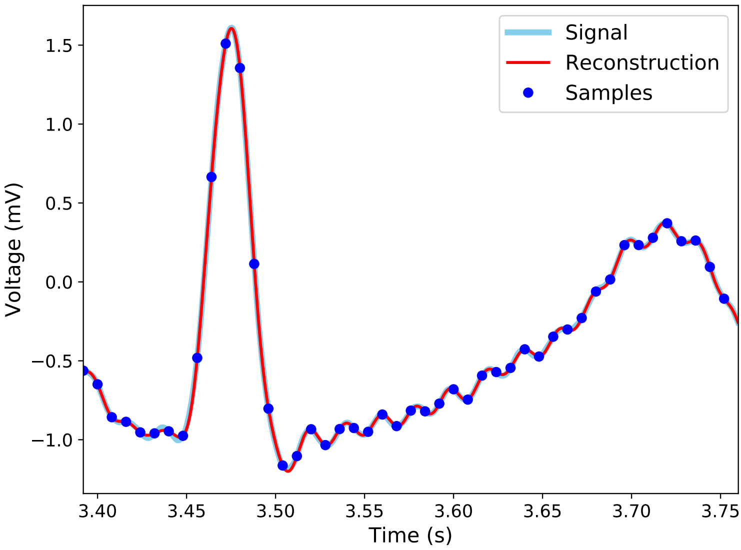
$T = 8$ s, so $k_c = 50T = 400$

To avoid aliasing $N \geq 801$

Recovered Fourier coefficients (N=1,000)



Recovered signal (N=1,000)



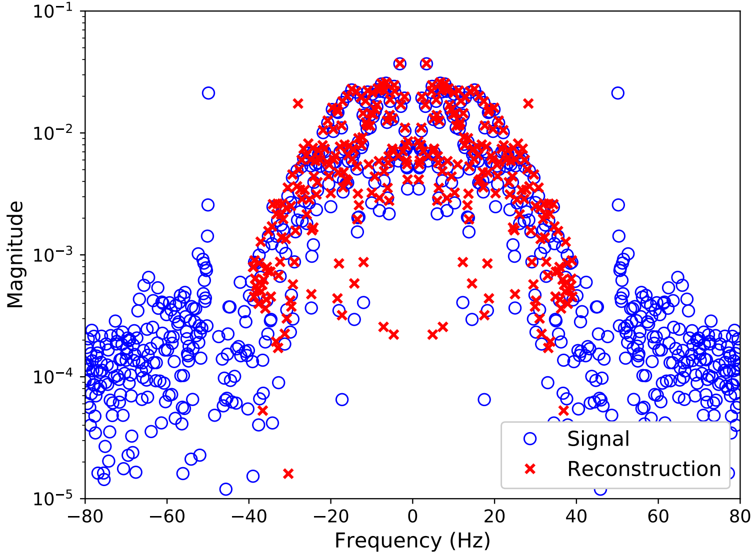
Sampling an electrocardiogram

We mistakenly assume that the signal is approximately bandlimited at around 40 Hz so $k_c = 312$ and $N = 625$

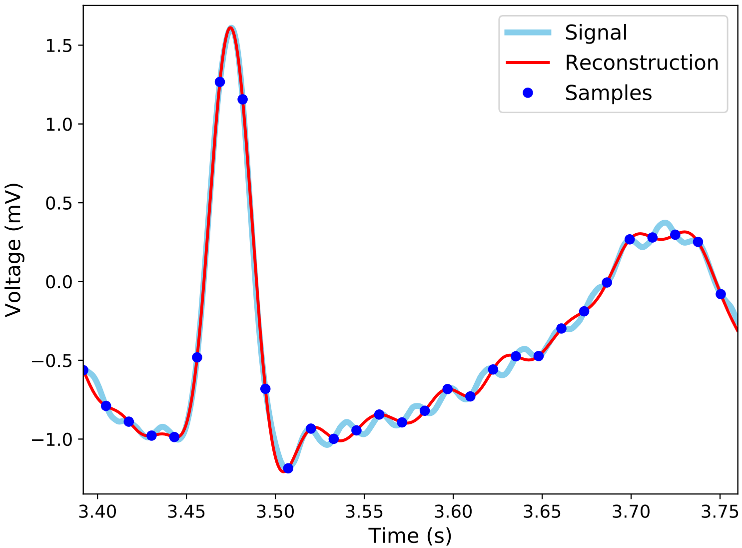
$$\hat{x}^{\text{rec}}[k] = \sum_{\{(m-k) \bmod 625 = 0\}} \hat{x}[m]$$

Component at $m = \pm 400$ (50 Hz) shows up at ± 225 (28.1 Hz)

Recovered Fourier coefficients ($N = 625$)



Recovered signal ($N = 625$)



What have we learned

Definition of orthogonal basis of discrete complex sinusoids

How to recover bandlimited signals from a finite number of samples

That insufficient sampling leads to aliasing