Subgradients

DS-GA 1013 / MATH-GA 2824 Mathematical Tools for Data Science

Carlos Fernandez-Granda

Prerequisites

Calculus (multivariate functions, gradients)

Linear algebra (norms)

Sparse regression via the lasso

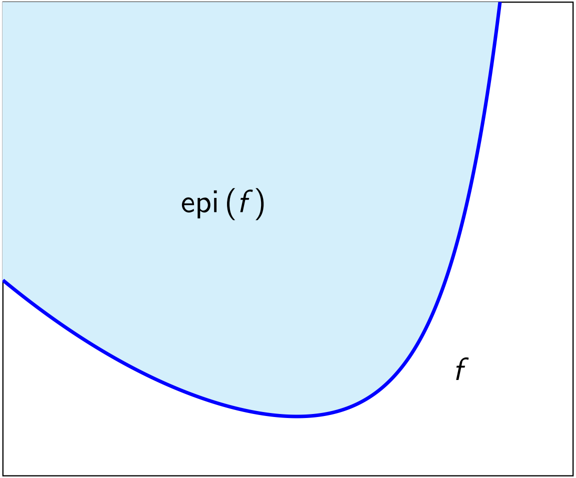
Convexity

Epigraph

The epigraph of $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is a set in $\mathbb{R}^{n+1}$

$$\operatorname{epi}(f) := \left\{ x \mid f\left(\begin{bmatrix} x[1] \\ \cdots \\ x[n] \end{bmatrix} \right) \leq x[n+1] \right\}$$

Epigraph

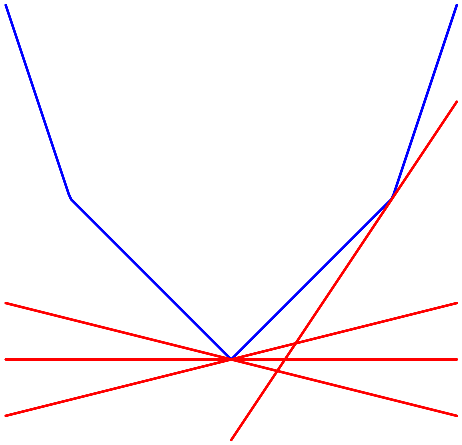


Supporting hyperplane

A hyperplane $\mathcal{H}$ is a supporting hyperplane of a set $\mathcal{S}$ at x if

- $\mathcal{H}$ and $\mathcal{S}$ intersect at x
- $\mathcal{S}$ is contained in one of the half-spaces bounded by $\mathcal{H}$

Supporting hyperplane



Convexity

A function $f : \mathbb{R}^n \to \mathbb{R}$ is convex if and only if its epigraph has a supporting hyperplane at every point

It is strictly convex if and only for all $x \in \mathbb{R}^n$ it only intersects with the supporting hyperplane at one point

Subgradients

The subgradient of $f : \mathbb{R}^n \to \mathbb{R}$ at $x \in \mathbb{R}^n$ is a vector $g \in \mathbb{R}^n$ such that

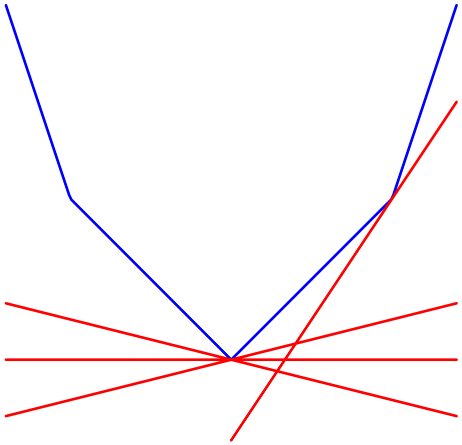
$$f(y) \geq f(x) + g^T (y - x), \quad \text{for all } y \in \mathbb{R}^n$$

The hyperplane

$$\mathcal{H}_g := \left\{ y \mid y[n+1] = f(x) + g^T \left(\begin{bmatrix} y[1] \\ \cdots \\ y[n] \end{bmatrix} - x \right) \right\}$$

is a supporting hyperplane of the epigraph of f at $\begin{bmatrix} x \\ f(x) \end{bmatrix}$

Subgradients



Subgradient of differentiable function

If a function is differentiable, the only subgradient at each point is the gradient

Proof

Assume g is a subgradient at x, for any $\alpha \geq 0$

$$\begin{aligned} f(x + \alpha e_i) &\geq f(x) + g^T \alpha e_i \\ &= f(x) + g[i]\,\alpha \\ f(x) &\leq f(x - \alpha e_i) + g^T \alpha e_i \\ &= f(x - \alpha e_i) + g[i]\,\alpha \end{aligned}$$

Combining both inequalities

$$\frac{f(x) - f(x - \alpha e_i)}{\alpha} \leq g[i] \leq \frac{f(x + \alpha e_i) - f(x)}{\alpha}$$

Letting $\alpha \to 0$, implies $g[i] = \frac{\partial f(x)}{\partial x[i]}$

Optimality condition for nondifferentiable functions

x is a minimum of f if and only if the zero vector is a subgradient of f at x

$$\begin{aligned} f(y) &\geq f(x) + \vec{0}^T (y - x) \\ &= f(x) \end{aligned}$$

for all $y \in \mathbb{R}^n$

Under strict convexity the minimum is unique

Sum of subgradients

Let g_1 and g_2 be subgradients at $x \in \mathbb{R}^n$ of $f_1 : \mathbb{R}^n \to \mathbb{R}$ and $f_2 : \mathbb{R}^n \to \mathbb{R}$

$g := g_1 + g_2$ is a subgradient of $f := f_1 + f_2$ at x

Proof: For any $y \in \mathbb{R}^n$

$$\begin{aligned} f(y) &= f_1(y) + f_2(y) \\ &\geq f_1(x) + g_1{}^T(y - x) + f_2(y) + g_2{}^T(y - x) \\ &\geq f(x) + g^T(y - x) \end{aligned}$$

Subgradient of scaled function

Let g_1 be a subgradient at $x \in \mathbb{R}^n$ of $f_1 : \mathbb{R}^n \to \mathbb{R}$

For any $\alpha \geq 0$ $g_2 := \alpha g_1$ is a subgradient of $f_2 := \alpha f_1$ at x

<u>Proof:</u> For any $y \in \mathbb{R}^n$

$$\begin{aligned} f_2(y) &= \alpha f_1(y) \\ &\geq \alpha \left(f_1(x) + {g_1}^T (y - x) \right) \\ &\geq f_2(x) + {g_2}^T (y - x) \end{aligned}$$

Subdifferential of absolute value

At $x \neq 0$, $f(x) = |x|$ is differentiable, so $g = \text{sign}(x)$

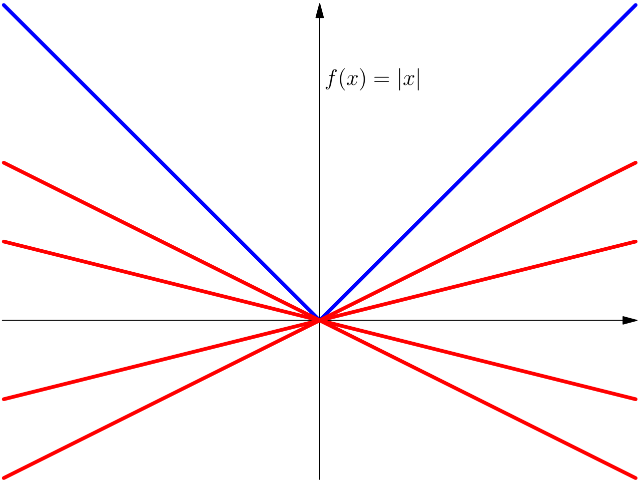
At $x = 0$, we need

$$f(0 + y) \geq f(0) + g(y - 0)$$

$$|y| \geq gy$$

Holds if and only if $|g| \leq 1$

Subdifferential of absolute value



Subdifferential of ℓ_1 norm

g is a subgradient of the ℓ_1 norm at $x \in \mathbb{R}^n$ if and only if

$$g[i] = \operatorname{sign}(x[i]) \qquad \text{if } x[i] \neq 0$$

$$|g[i]| \leq 1 \qquad \text{if } x[i] = 0$$

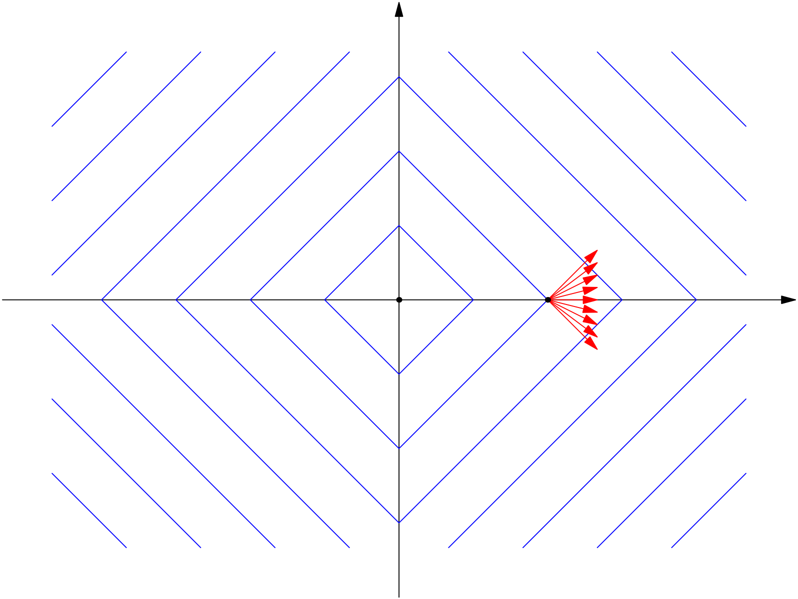
Proof (one direction)

Assume $g[i]$ is a subgradient of $|\cdot|$ at $|x[i]|$ for $1 \leq i \leq n$

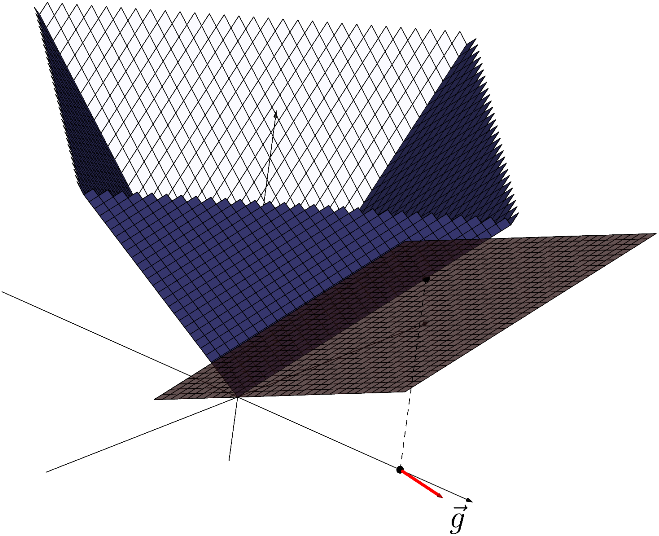
For any $y \in \mathbb{R}^n$

$$
\begin{aligned}
||y||_1 &= \sum_{i=1}^{n} |y[i]| \\
&\geq \sum_{i=1}^{n} |x[i]| + g[i]\,(y[i] - x[i]) \\
&= ||x||_1 + g^T (y - x)
\end{aligned}
$$

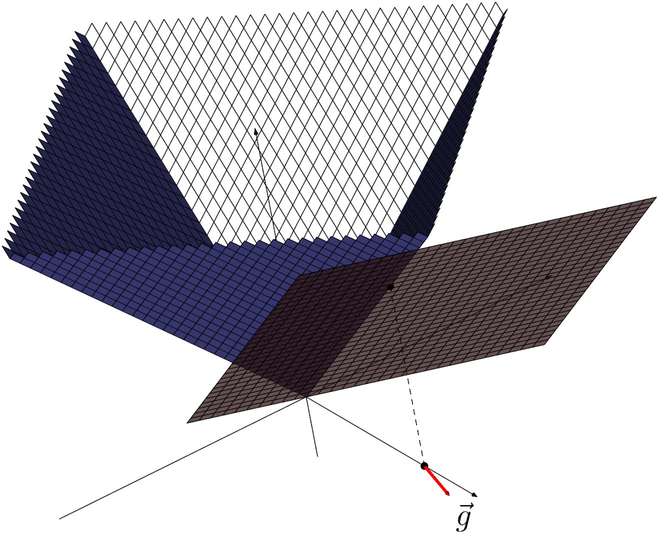
Subdifferential of ℓ_1 norm



Subdifferential of ℓ_1 norm



Subdifferential of ℓ_1 norm



What have we learned?

Definition of subgradients

Optimality condition for nondifferentiable convex functions

Subgradients of ℓ_1 norm