



Time-frequency analysis (blended lecture)

DS-GA 1013 / MATH-GA 2824 Mathematical Tools for Data Science

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Amplitude modulation

Short-time Fourier transform

Multiplying by a real sinusoid

What happens if we multiply an arbitrary signal x and a sinusoid?

$$s[j] := \cos\left(\frac{2\pi k^* j}{N}\right) \quad 0 \leq j \leq N$$

$$\widehat{x_s} =$$

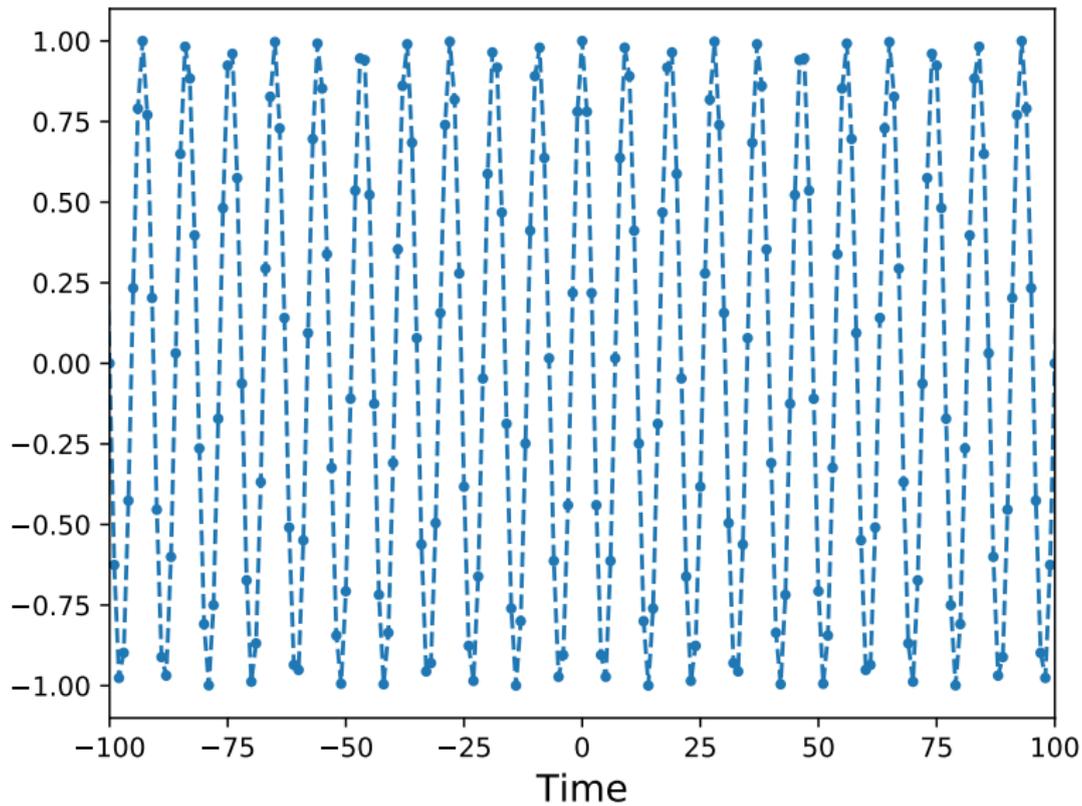
Multiplying by a real sinusoid

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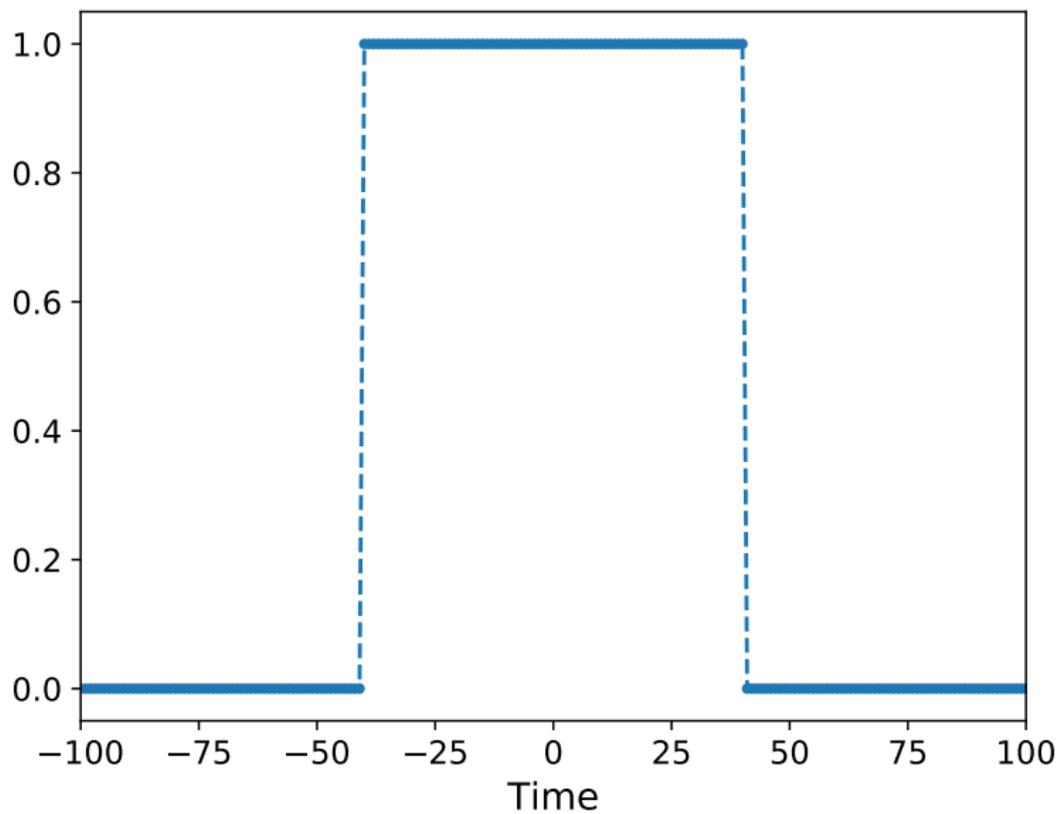
$$s[j] := \cos\left(\frac{2\pi k^* j}{N}\right) \quad 0 \leq j \leq N$$

$$\begin{aligned}\hat{x}s &= \sum_{j=1}^N x[j] \psi_{k^*}[j] s[j] \exp\left(-\frac{i2\pi kj}{N}\right) \\ &= \sum_{j=1}^N x[j] \frac{\exp\left(\frac{i2\pi k^* j}{N}\right) + \exp\left(\frac{-i2\pi k^* j}{N}\right)}{2} \exp\left(-\frac{i2\pi kj}{N}\right) \\ &= \frac{1}{2} \sum_{j=1}^N x[j] \exp\left(\frac{-i2\pi(k - k^*)j}{N}\right) + \frac{1}{2} \sum_{j=1}^N x[j] \exp\left(\frac{-i2\pi(k + k^*)j}{N}\right) \\ &= \frac{\hat{x}[(k - k^*)] + \hat{x}[k + k^*]}{2}\end{aligned}$$

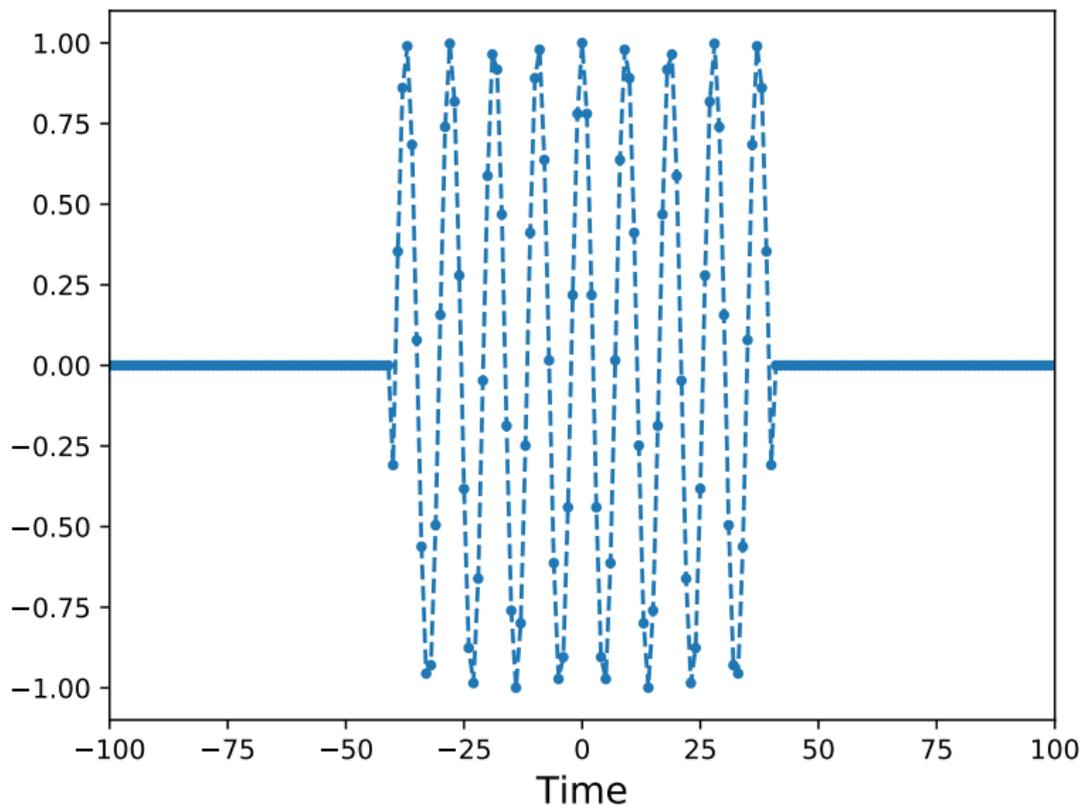
Signal



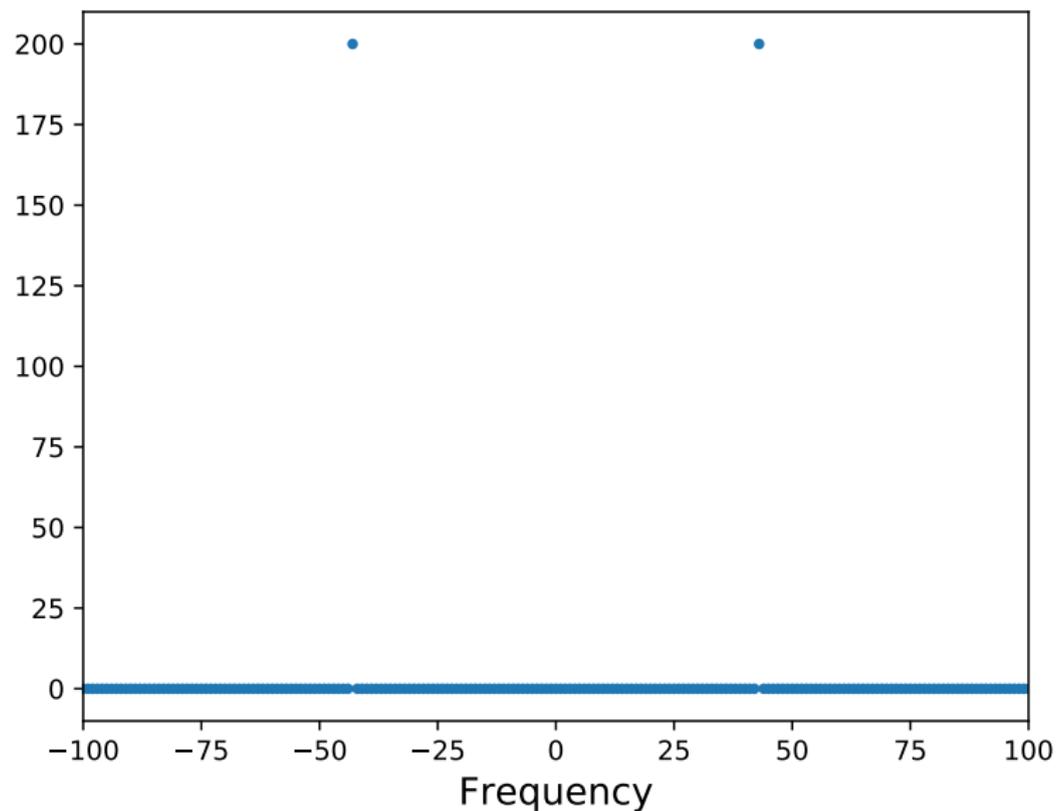
Window



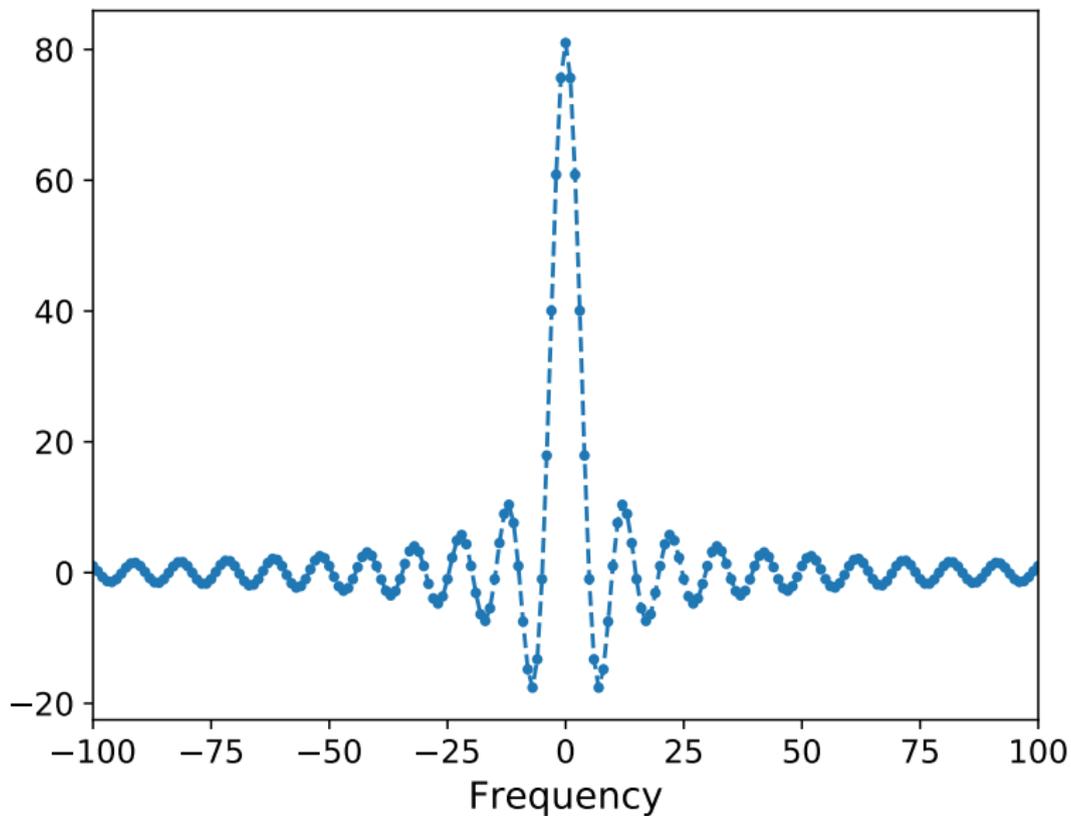
Windowed signal



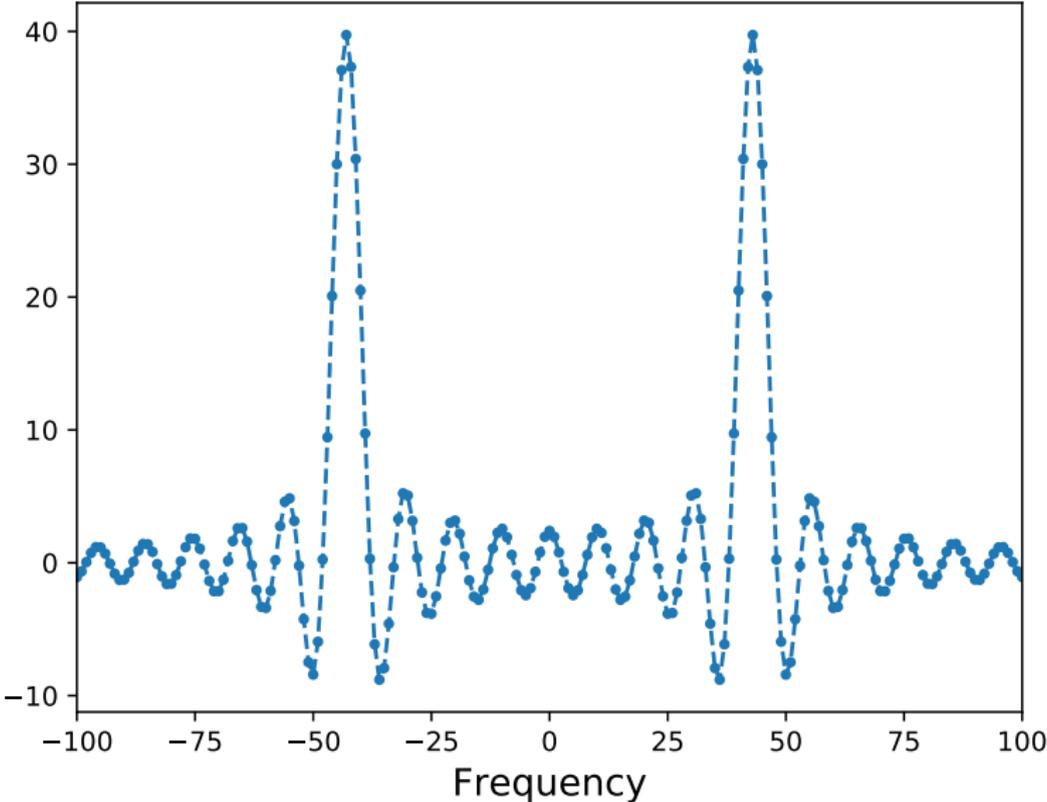
DFT of signal (are we going to see something like this?)



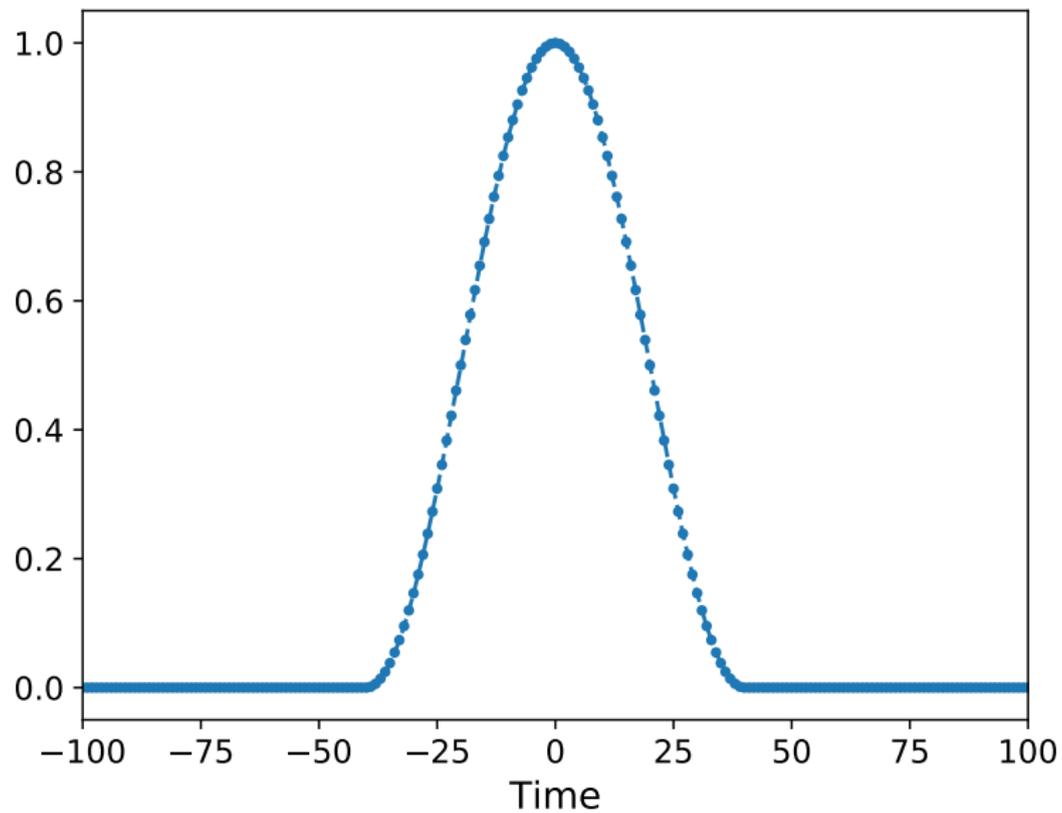
DFT of rectangular window



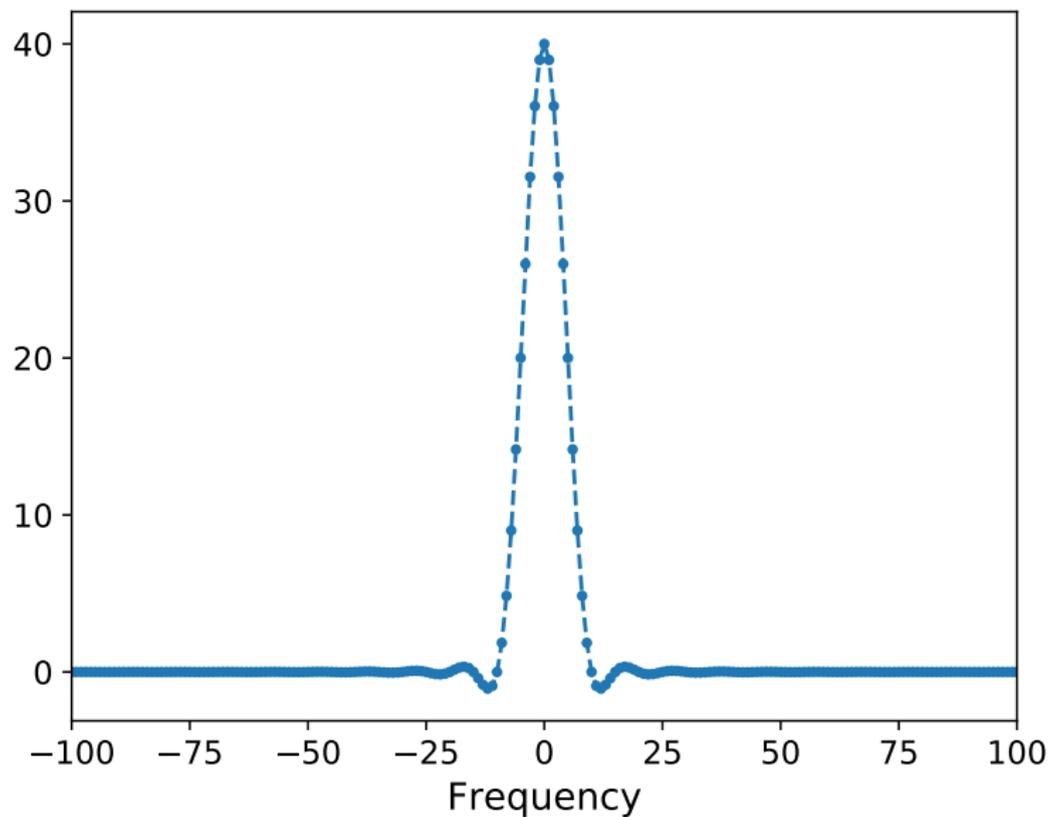
DFT of windowed signal



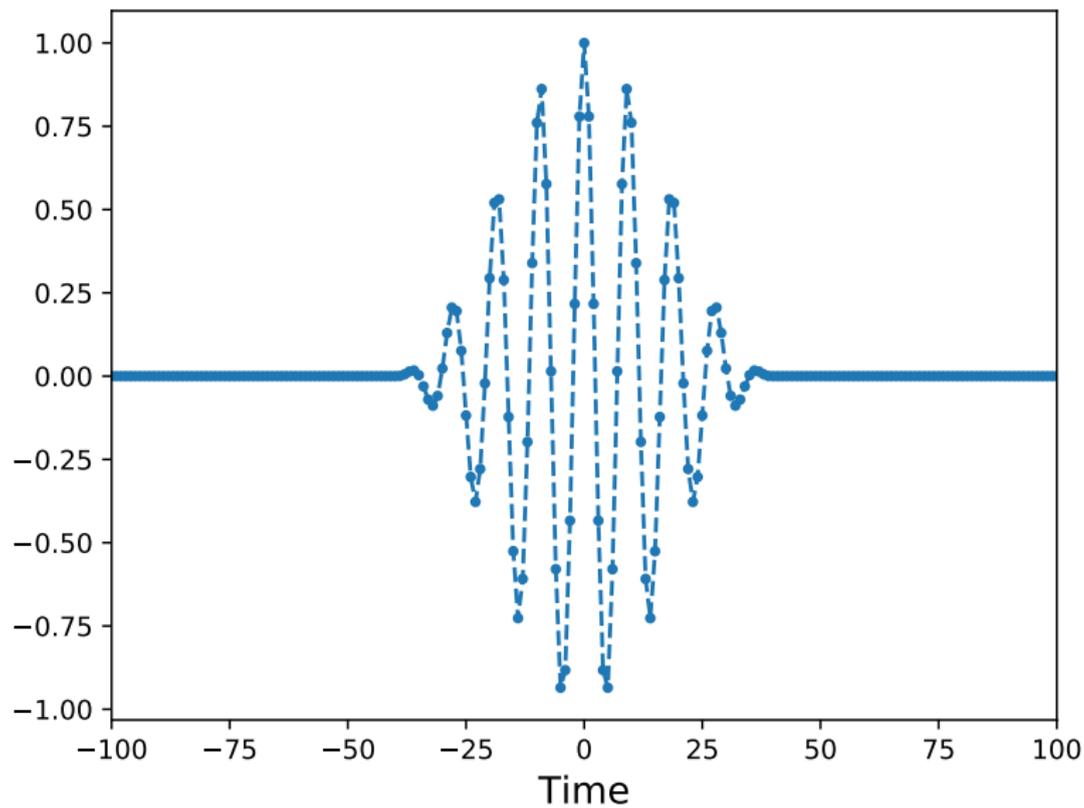
Hann window



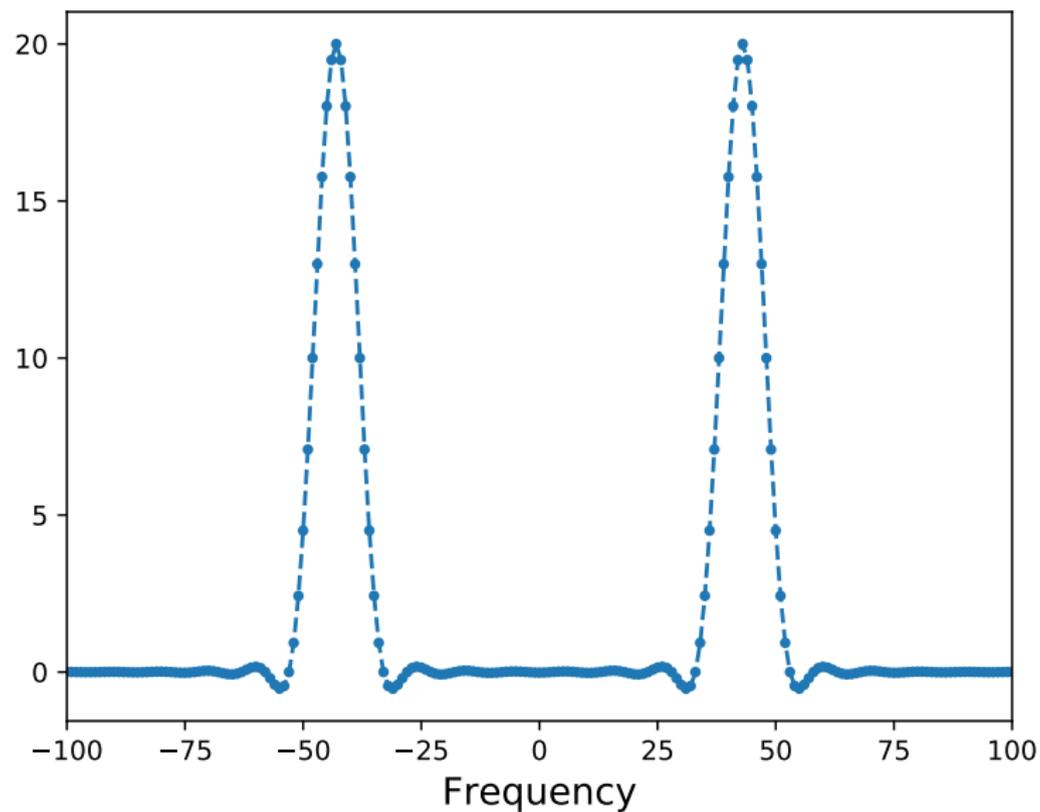
DFT of Hann window



Windowed signal



DFT of windowed signal



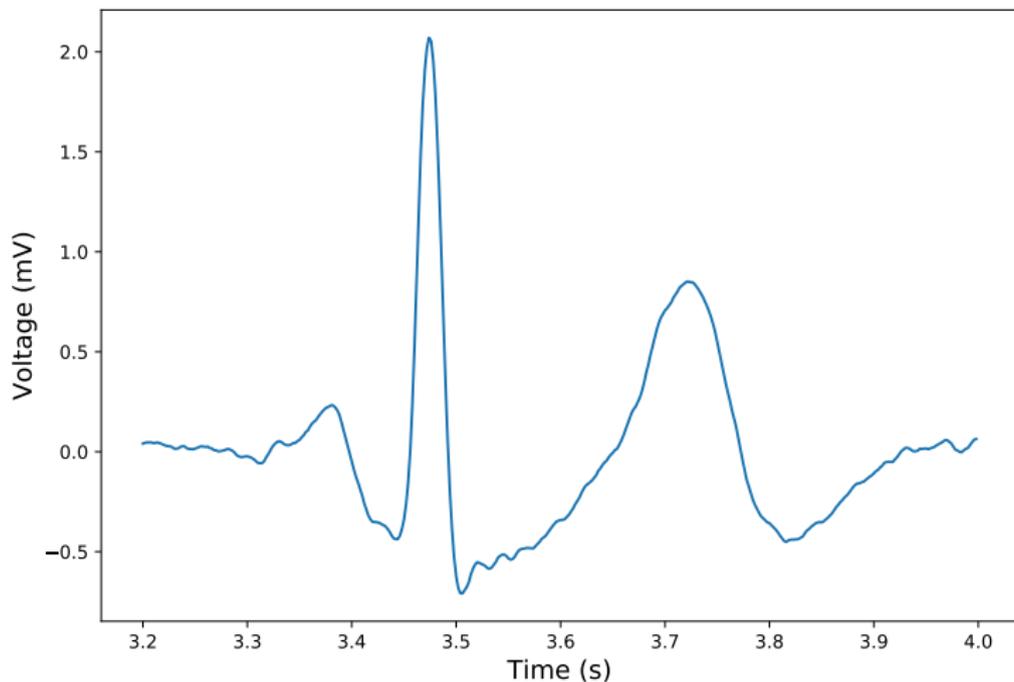
Amplitude modulation

Earliest method for transmitting audio in radio broadcasting (1900 by Reginald Fessenden)

Still in use!

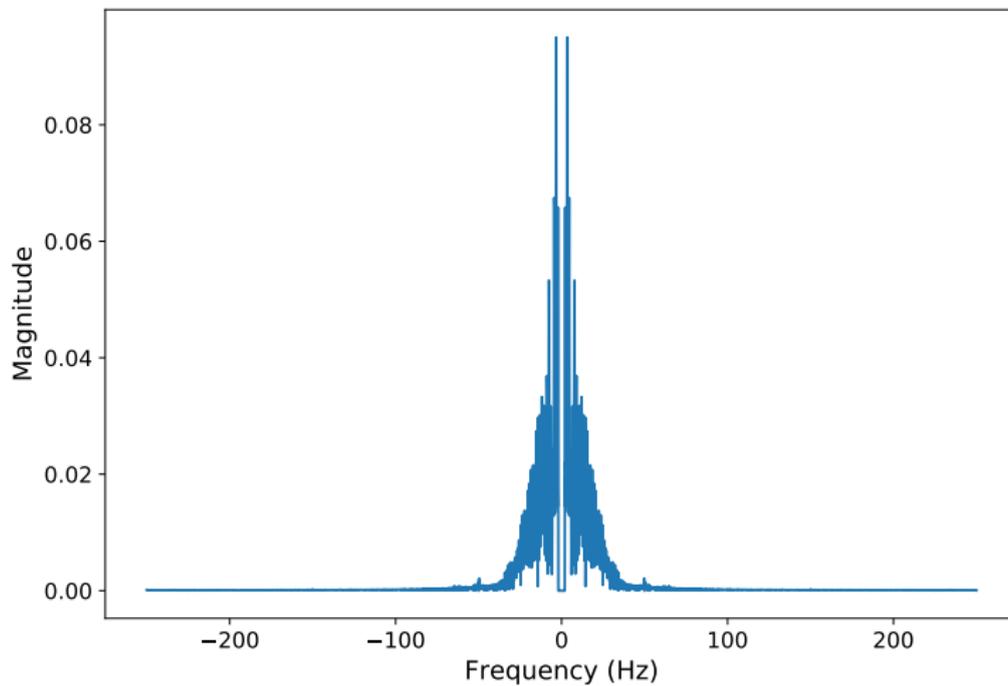
Similar ideas are the foundation for wireless communications (radio, mobile phones, satellites)

Electrocardiogram signal

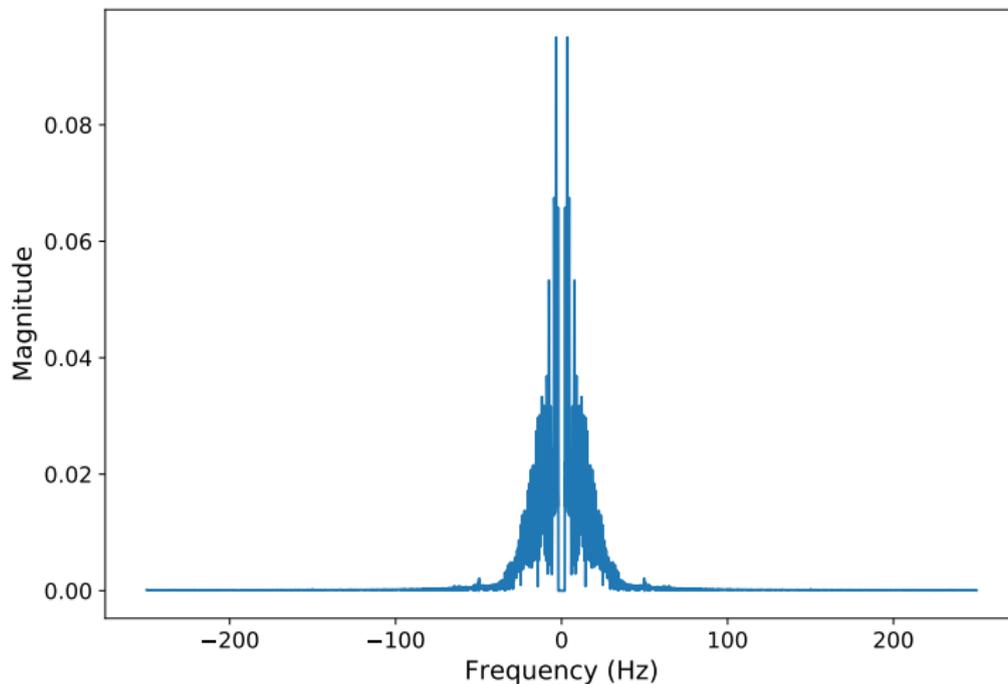


Challenge: How to transmit many such signals *at the same time*

Electrocardiogram signal



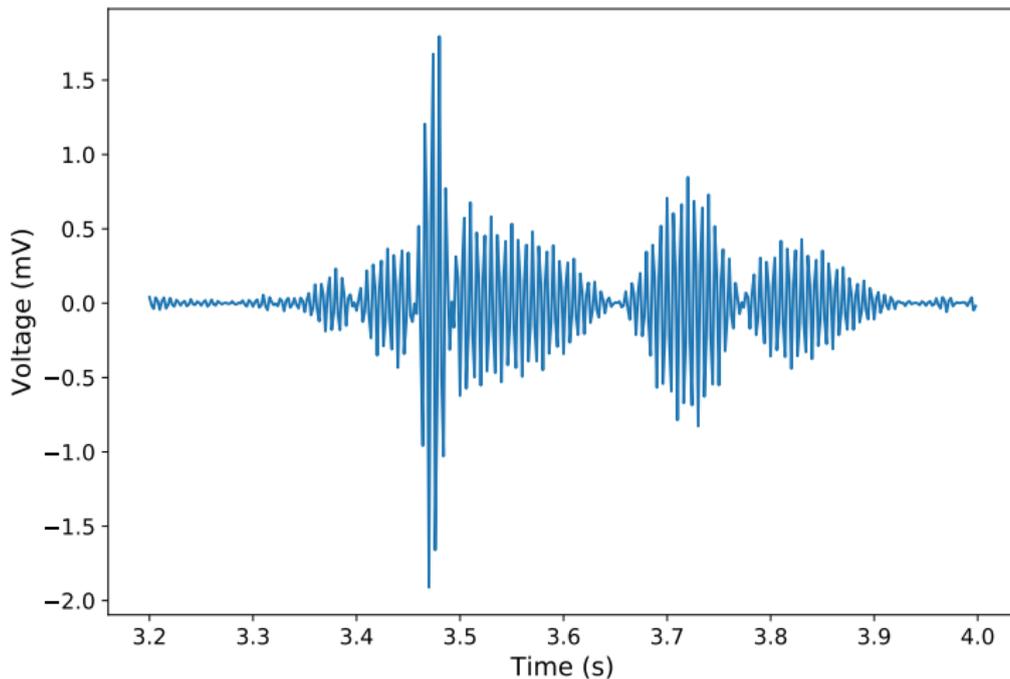
Electrocardiogram signal



Idea: *Shift* in frequency then combine

How do we shift in frequency?

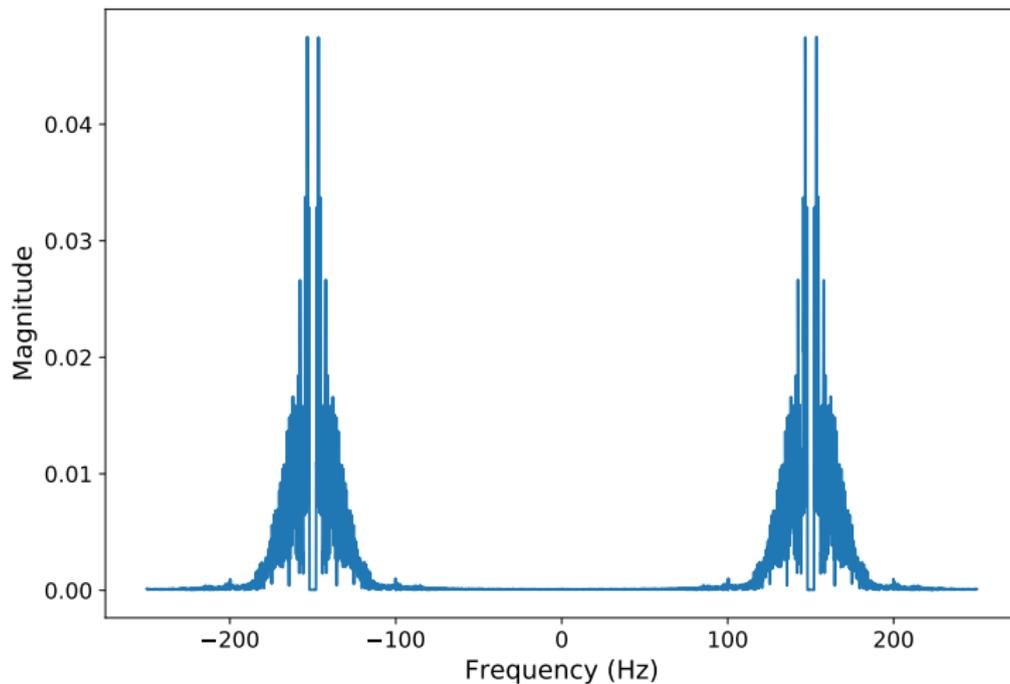
How do we shift in frequency? Multiply by sinusoid x_s !



Why is it called amplitude modulation?

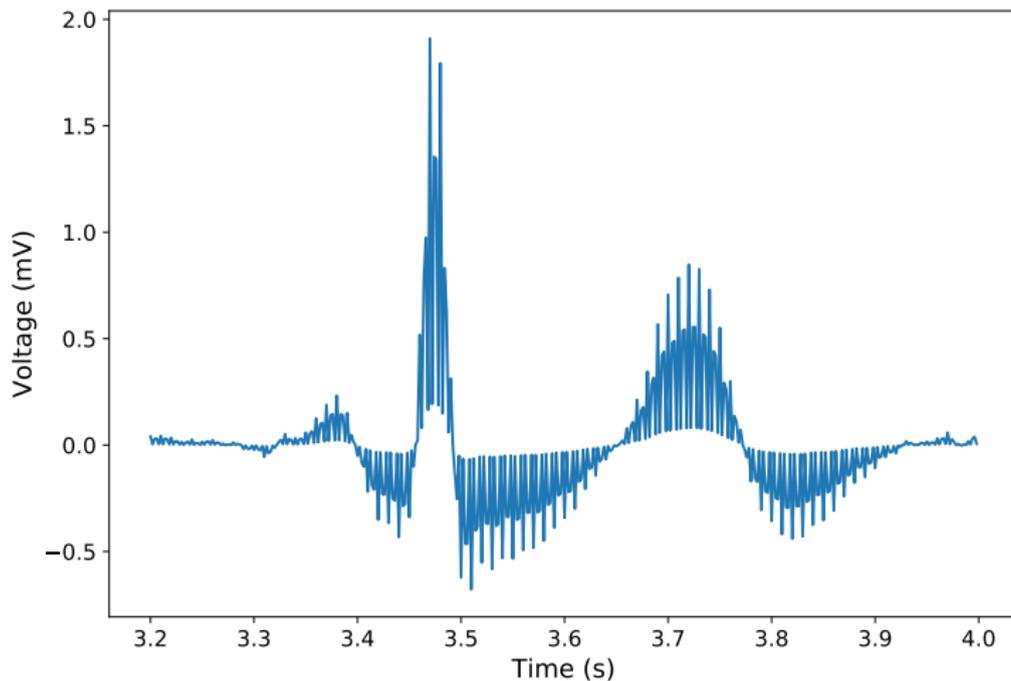
What happens in frequency domain?

What happens in frequency domain?



How do we recover the signal?

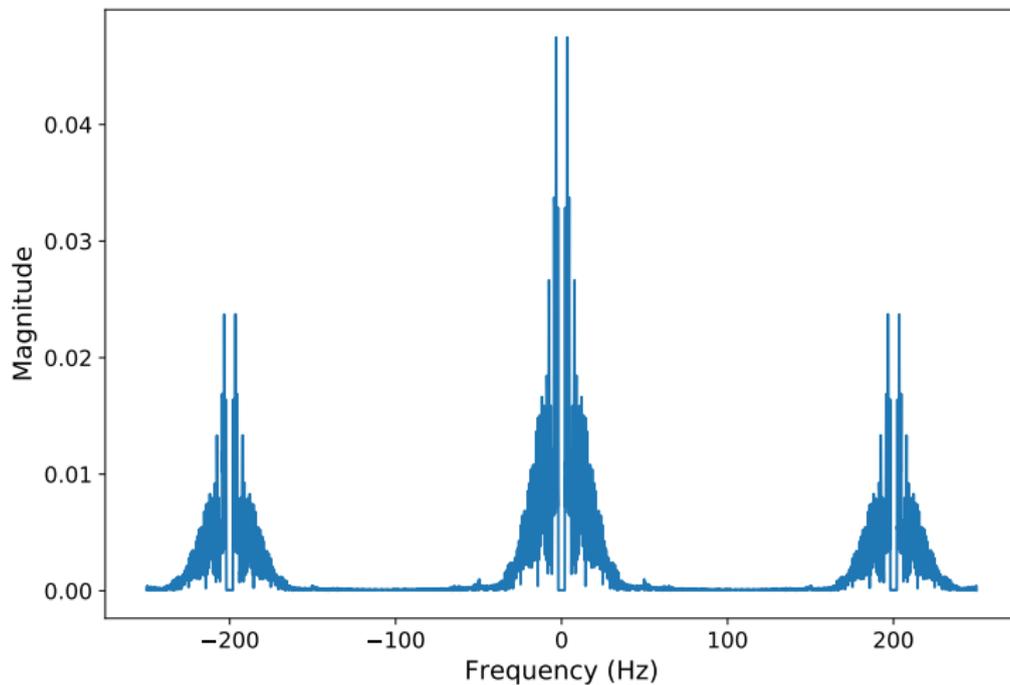
How do we recover the signal?



We start by multiplying by same sinusoid

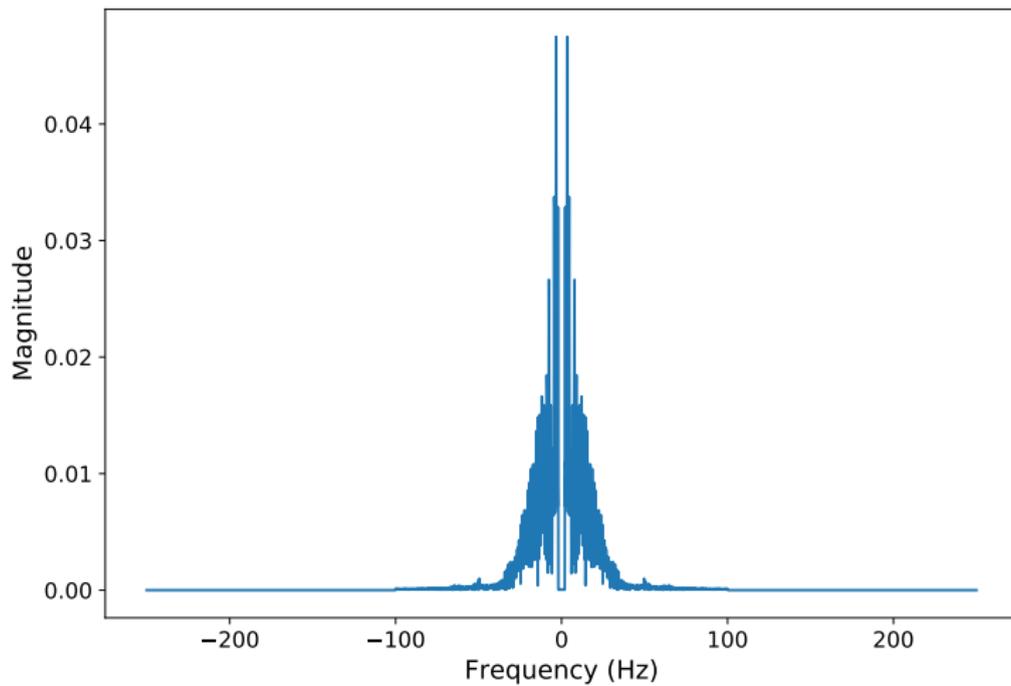
What happens in frequency domain?

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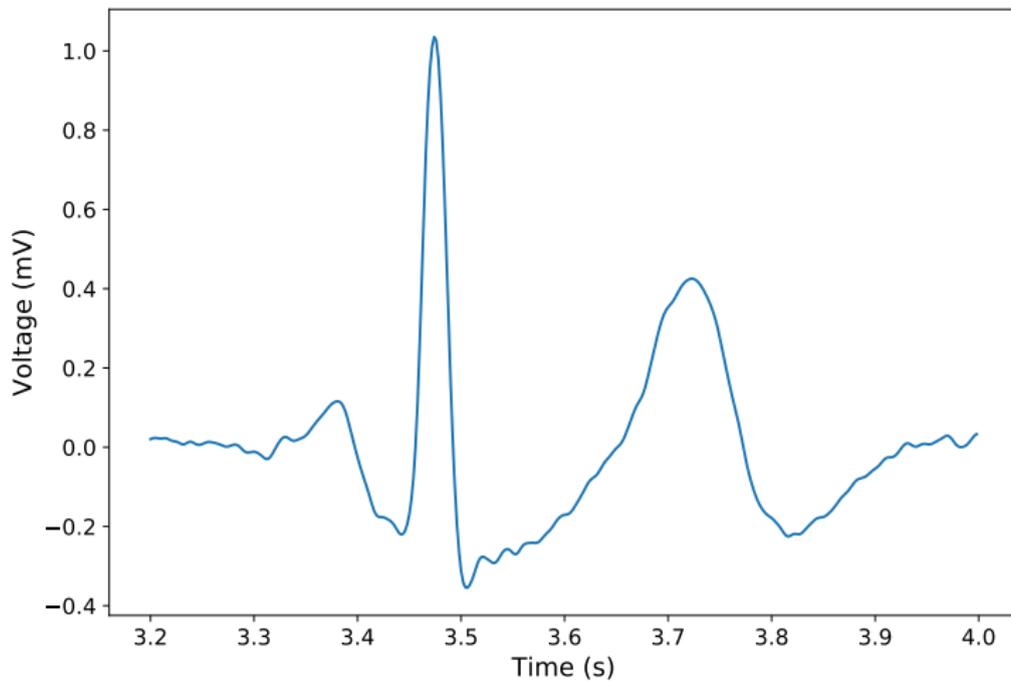


How do we recover the signal?

Filtered signal



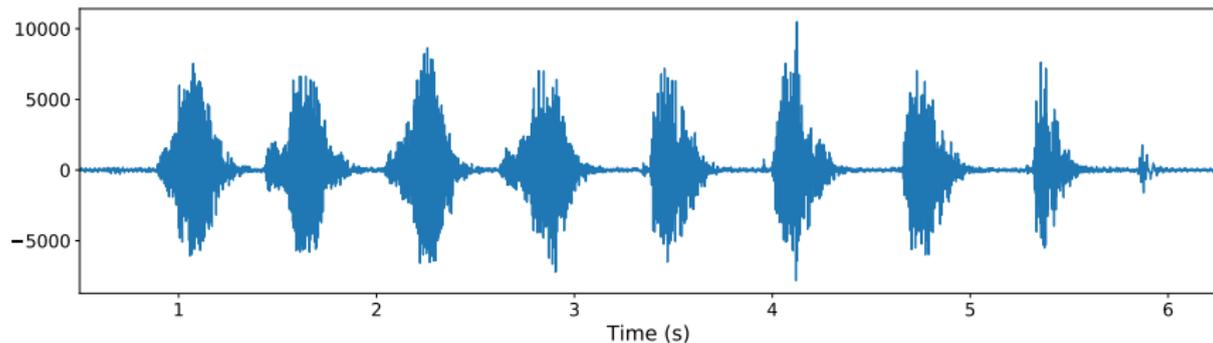
Filtered signal



Amplitude modulation

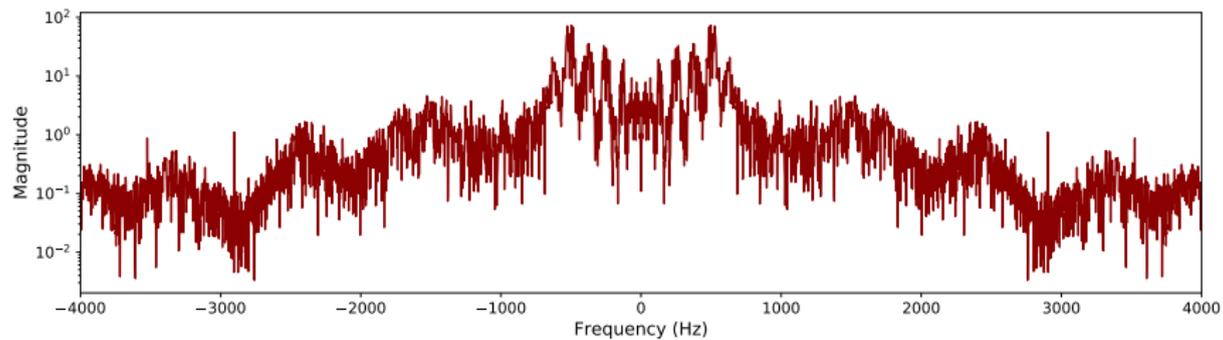
Short-time Fourier transform

Speech signal

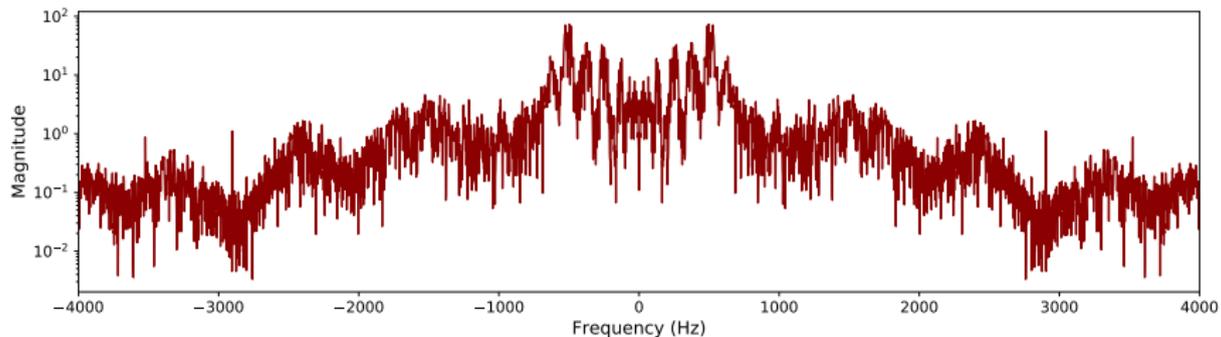


Challenge: Characterize how frequency components change *over time*

Fourier series



Fourier series



First segment or *window* signal, then compute Fourier series / DFT

Short-time Fourier transform

1. Segment in overlapping intervals of length ℓ
2. Multiply by window vector
3. Compute DFT of length ℓ

Short-time Fourier transform

The short-time Fourier transform of $x \in \mathbb{C}^N$ is

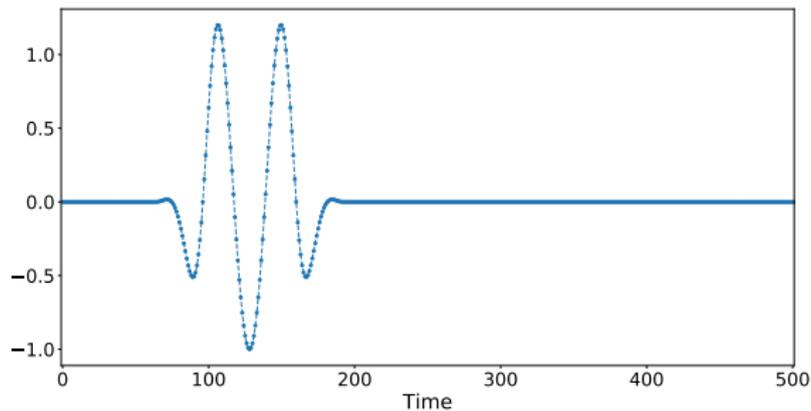
$$\text{STFT}_{[\ell]}(x)[k, s] := \left\langle x, \xi_k^{\downarrow s(1-\alpha_{\text{ov}})\ell} \right\rangle, \quad 0 \leq k \leq \ell - 1, \quad 0 \leq s \leq \frac{N-1}{(1-\alpha_{\text{ov}})\ell},$$

$$\xi_k[j] := \begin{cases} w_{[\ell]}[j] \exp\left(\frac{i2\pi kj}{\ell}\right) & \text{if } 1 \leq j \leq \ell \\ 0 & \text{otherwise} \end{cases}$$

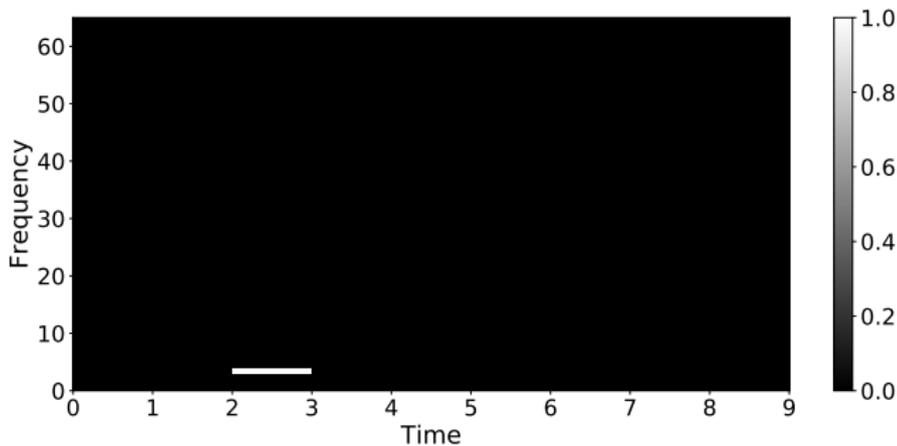
Overlap between adjacent segments equals $\alpha_{\text{ov}}\ell$

$k = 3$ $s = 2$ ($N := 500$, $\ell := 128$, $\alpha_{ov} := 0.5$)

Basis vector

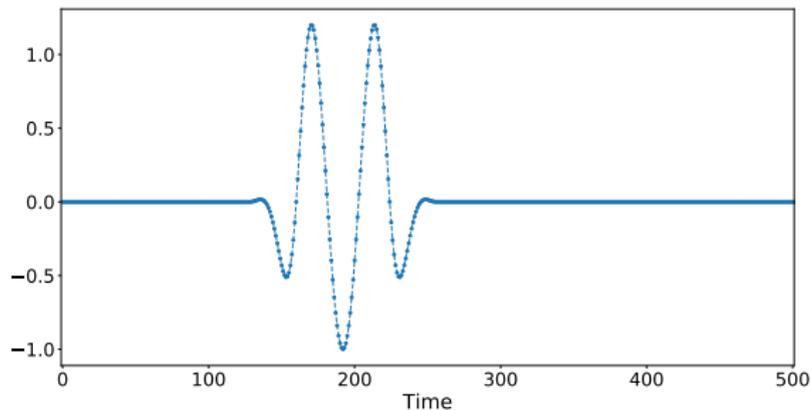


STFT
coefficient

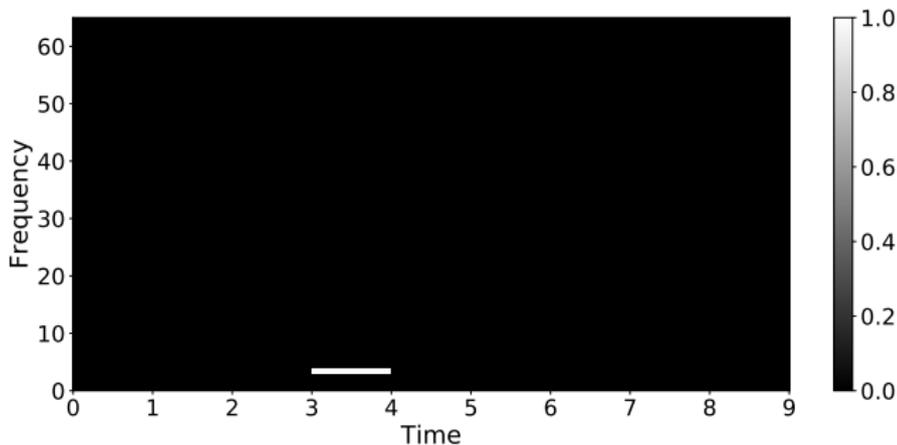


$k = 3$ $s = 3$ ($N := 500$, $\ell := 128$, $\alpha_{ov} := 0.5$)

Basis vector

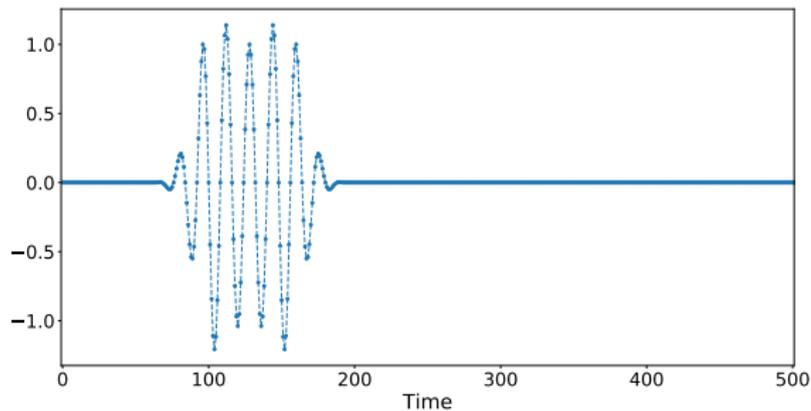


STFT
coefficient

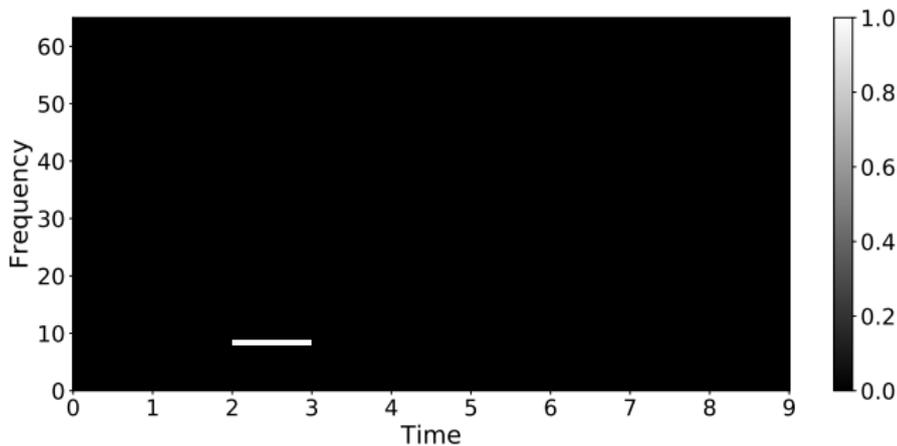


$k = 8$ $s = 2$ ($N := 500$, $\ell := 128$, $\alpha_{ov} := 0.5$)

Basis vector

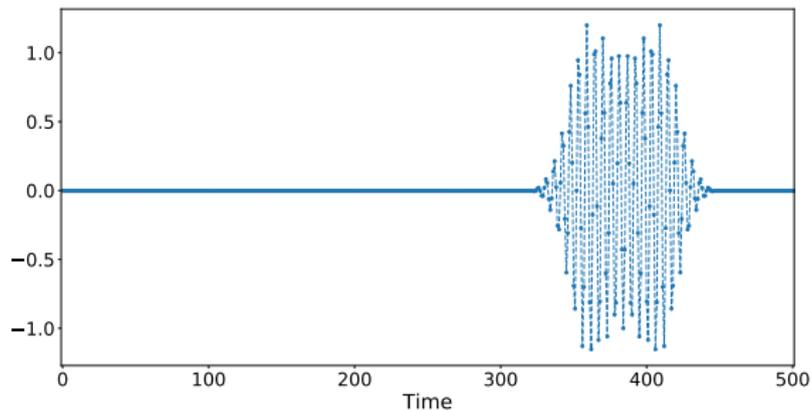


STFT
coefficient

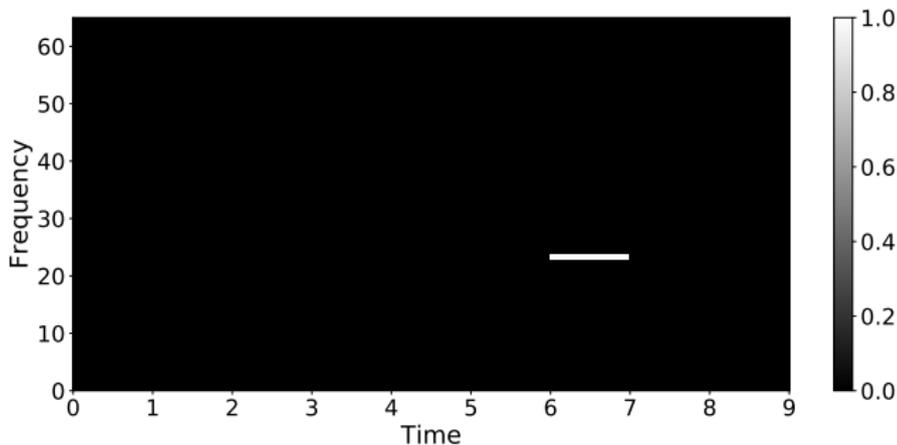


$k = 23$ $s = 6$ ($N := 500$, $\ell := 128$, $\alpha_{ov} := 0.5$)

Basis vector



STFT
coefficient



Matrix representation of STFT

Matrix representation of STFT

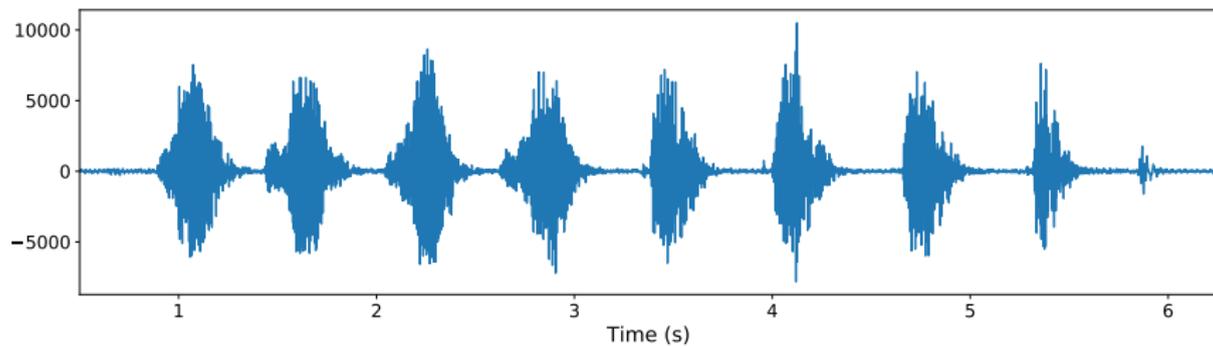
$$\begin{bmatrix} F_{[\ell]} & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & F_{[\ell]} & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & F_{[\ell]} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix} \begin{bmatrix} \text{diag}(w_{[\ell]}) & 0 & 0 & \cdots \\ 0 & \text{diag}(w_{[\ell]}) & 0 & \cdots \\ 0 & 0 & \text{diag}(w_{[\ell]}) & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix} \mathbf{x}$$

Inverting the STFT

Apply inverse DFT to each segment

Combine segments

Speech signal



Spectrogram (window length = 62.5 ms)

