Wavelets

DS-GA 1013 / MATH-GA 2824 Mathematical Tools for Data Science

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Prerequisites

Linear algebra (basis, projection, orthogonal complement, direct sum)

Fourier series

Discrete Fourier transform

Short-time Fourier transform
Vertical line (column 135)
Multiresolution analysis

Scale / resolution at which information is encoded is not uniform

Goal: Decompose signals into components at different resolutions

Challenge: Design basis of vectors to achieve this

If vectors are orthogonal, then we can just project onto them to separate contributions of each scale
Father wavelet

We use a low-pass vector, called scaling vector or father wavelet, to extract coarsest scale.

Haar father wavelet
Approximation using Haar father wavelet

Approximation

Coefficients
Mother wavelets

We use shifts and dilations of mother wavelet to capture information at different scales.

We can choose the shifts so that the basis vectors are all orthogonal.
Father wavelet + coarsest mother wavelet

Approximation

Coefficients

Data

Approximation

0 100 200 300 400 500

0.2

0.4

0.6

0.8

1.0

Data

Approximation

0.04

0.02

0.00

0.02

0.04

0.0

0.5

1.0

1.5

2.0

2.5

3.0

3.5

4.0

0.0

0.04

0.02

0.00

0.02

0.04
Father wavelet + 2 coarsest mother wavelets

Approximation

Coefficients

Data

Approximation

0 100 200 300 400 500

0.2

0.4

0.6

0.8

1.0

0.0

0.2

0.4

0.6

0.8

0.0 0.2 0.4 0.6 0.8 1.0
Father wavelet + 3 coarsest mother wavelets

Approximation

Coefficients
Father wavelet + 4 coarsest mother wavelets
Father wavelet + 5 coarsest mother wavelets

Approximation

Coefficients

Data

Approximation
Father wavelet + 6 coarsest mother wavelets

Approximation

Coefficients
Father wavelet + 7 coarsest mother wavelets

Approximation

Coefficients
Father wavelet + 8 coarsest mother wavelets

Approximation

Coefficients
Father wavelet + 9 coarsest mother wavelets

Approximation

Coefficients
Let $N := 2^K$ for some $K$, a multiresolution decomposition of $\mathbb{R}^N$ is a sequence of nested subspaces $\mathcal{V}_K \subset \mathcal{V}_{K-1} \subset \ldots \subset \mathcal{V}_0$ satisfying:

- $\mathcal{V}_0 = \mathbb{R}^N$

- If $x \in \mathcal{V}_k$ then $x$ shifted by $2^k$ is also in $\mathcal{V}_k$ (invariance to translations)

- Dilating $x \in \mathcal{V}_j$ yields vector in $\mathcal{V}_{j+1}$
Example

Subspace $V_k$ contains vectors that are constant on segments of length $2^k$

Satisfies conditions:

- $V_0 = \mathbb{R}^N$
- If $x \in V_k$ then $x$ shifted by $2^k$ is also in $V_k$ (invariance to translations)
- Dilating $x \in V_j$ yields vector in $V_{j+1}$

Spanned by shifts/dilations of Haar father wavelets

Problem: Basis vectors are not orthogonal (at all!)
Solution

Decompose the finer subspaces into a direct sum

$$V_k = V_{k+1} \oplus W_k, \quad 0 \leq k \leq K - 1,$$

$W_k$ is the orthogonal complement of $V_{k+1}$ in $V_k$, so it captures finest resolution available at level $k$

We can then decompose $\mathbb{R}^N$ into different scales

$$\mathbb{R}^N = V_0 = V_1 \oplus W_1$$
$$= V_2 \oplus W_2 \oplus W_1$$
$$= V_k \oplus W_k \oplus \cdots \oplus W_2 \oplus W_1$$
Haar multiresolution decomposition

\[ \mathcal{P}_{\mathcal{V}_k} x \text{ is an approximation of } x \text{ at scale } 2^k \]
Vertical line (column 135)
Projection onto $\mathcal{V}_9$

**Projection**

- **Data**
- **Approximation**

**Coefficients for $\mathcal{V}_9$**

- $x$-axis: $-0.04$ to $0.04$
- $y$-axis: $0$ to $16$
Projection onto $\mathcal{V}_8$

Projection

Coefficients for $\mathcal{W}_9$
Projection onto $\mathcal{V}_7$

![Graph showing Projection and Coefficients for $\mathcal{W}_8$]
Projection onto $\mathcal{V}_6$

**Projection**

Data

Approximation

**Coefficients for $\mathcal{W}_7$**
Projection onto $\mathcal{V}_5$

Projection

Coefficients for $\mathcal{W}_6$
Projection onto $\mathcal{V}_4$

Projection

Coefficients for $\mathcal{W}_5$
Projection onto $\mathcal{V}_3$

Projection

Coefficients for $\mathcal{W}_4$
Projection onto $\mathcal{V}_2$

**Projection**

**Coefficients for $\mathcal{W}_3$**
Projection onto $\mathcal{V}_1$

Projection

Coefficients for $\mathcal{W}_2$
Projection onto $\mathcal{V}_0$

![Graph showing data and approximation](image)

**Projection**

**Coefficients for $\mathcal{W}_1$**

![Graph showing coefficients](image)
Haar mother wavelets in the frequency domain
Time-frequency support of basis vectors

STFT

Wavelets
2D Wavelets

Extension to 2D by using outer products of 1D basis vectors

To build a 2D basis vector at scale \((m_1, m_2)\) and shift \((s_1, s_2)\) we set

\[
\psi^{2D}_{[s_1, s_2, m_1, m_2]} := \psi^{1D}_{[s_1, m_1]} (\psi^{1D}_{[s_2, m_2]})^T,
\]

where \(\psi^{1D}\) can refer to 1D father or mother wavelets

Nonseparable designs: steerable pyramid, curvelets, bandlets...
2D Haar wavelet basis vectors
2D Haar wavelet decomposition
2D Haar wavelet decomposition

Approximation

Coefficients
2D Haar wavelet decomposition

Approximation

Coefficients
2D Haar wavelet decomposition

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Approximation

Coefficients
2D Haar wavelet decomposition
What have we learned

Framework for multiresolution analysis based on wavelets

Implementation based on Haar wavelets

Extension to two dimensions