Wavelets

DS-GA 1013 / MATH-GA 2824 Mathematical Tools for Data Science

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Prerequisites

Linear algebra (basis, projection, orthogonal complement, direct sum)

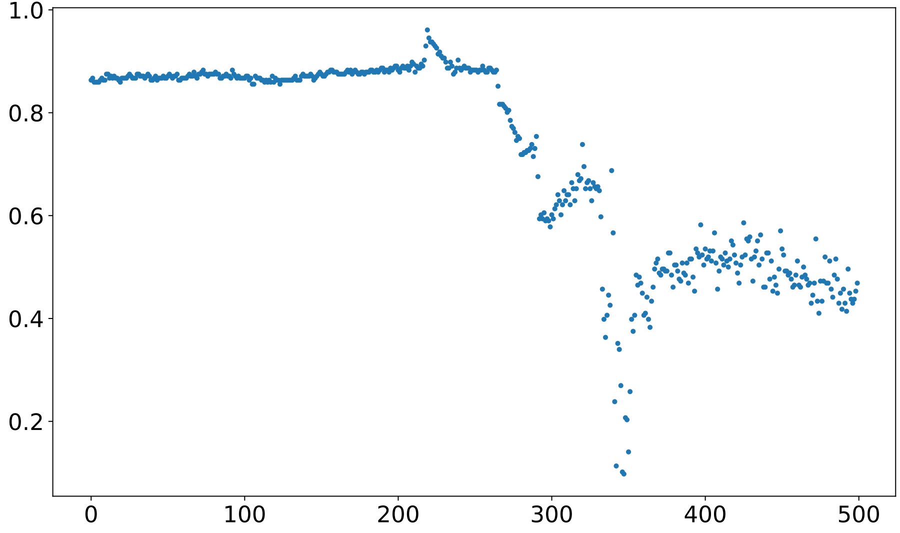
Fourier series

Discrete Fourier transform

Short-time Fourier transform

Image

Vertical line (column 135)



Multiresolution analysis

Scale / resolution at which information is encoded is not uniform

Goal: Decompose signals into components at different resolutions

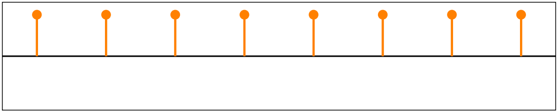
Challenge: Design basis of vectors to achieve this

If vectors are orthogonal, then we can just project onto them to separate contributions of each scale

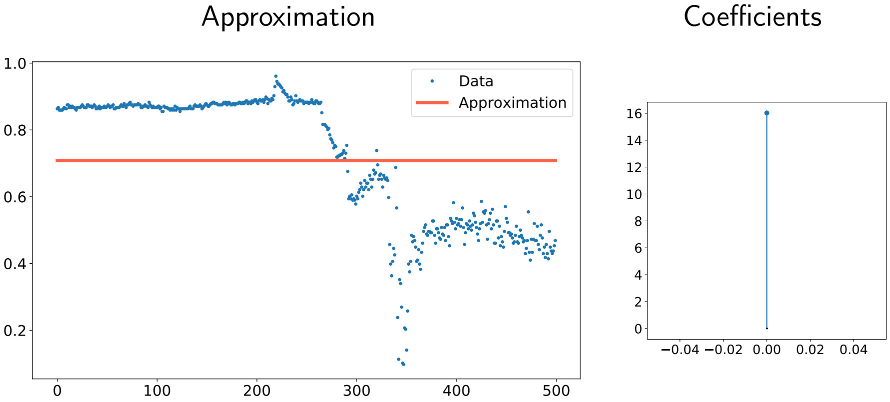
Father wavelet

We use a low-pass vector, called scaling vector or father wavelet, to extract coarsest scale

Haar father wavelet



Approximation using Haar father wavelet

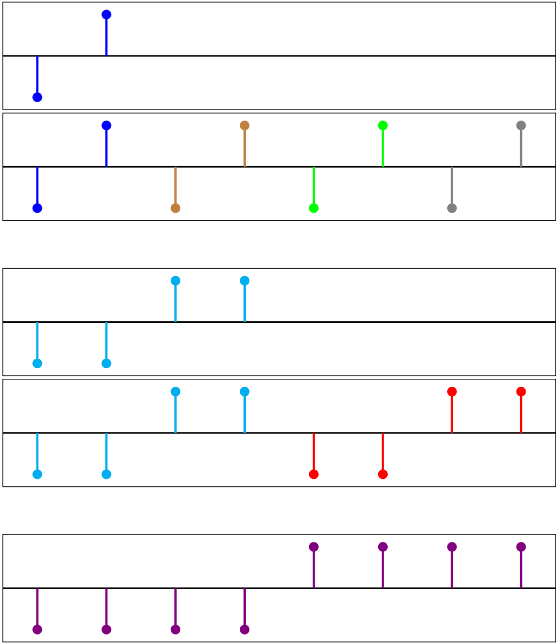


Mother wavelets

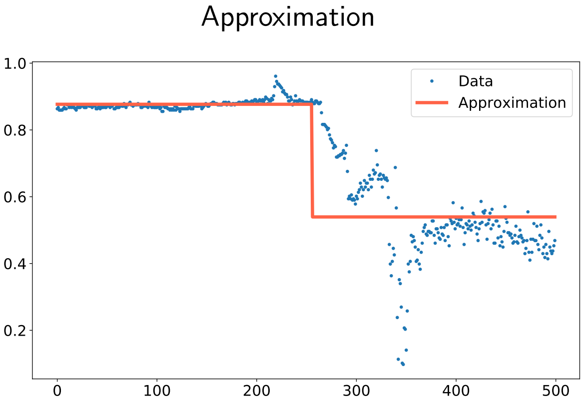
We use shifts and dilations of mother wavelet to capture information at different scales

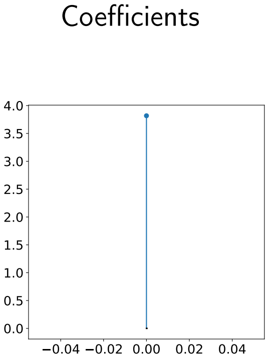
We can choose the shifts so that the basis vectors are all orthogonal

Haar mother wavelets

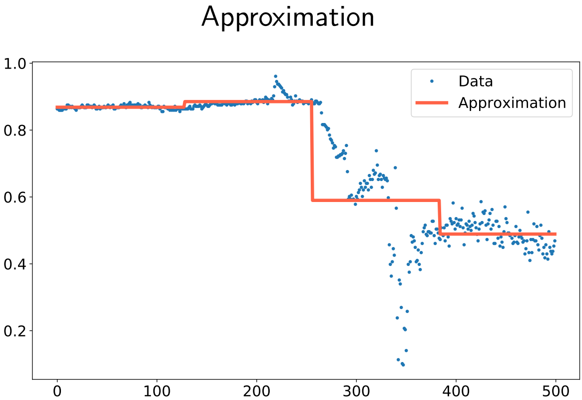


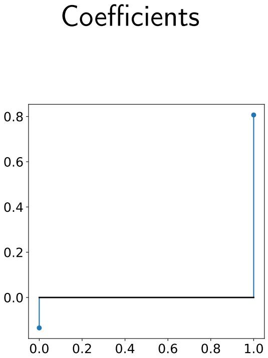
Father wavelet + coarsest mother wavelet



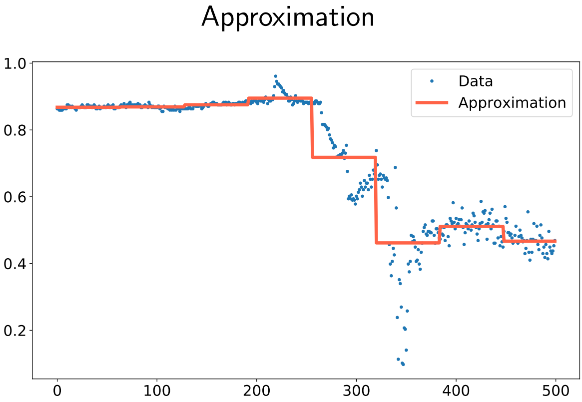


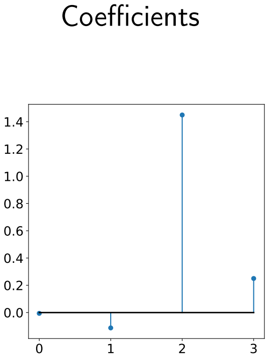
Father wavelet + 2 coarsest mother wavelets





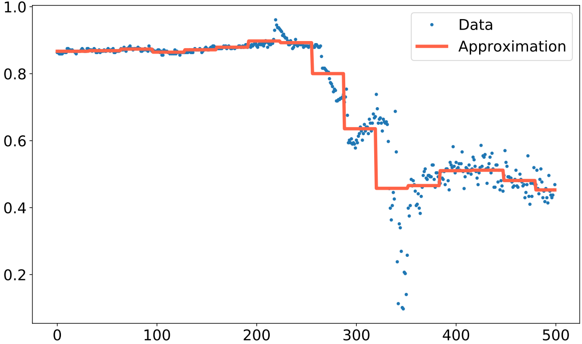
Father wavelet + 3 coarsest mother wavelets



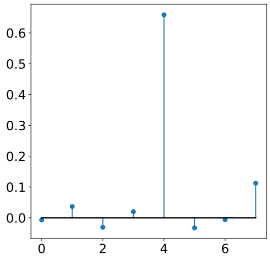


Father wavelet + 4 coarsest mother wavelets

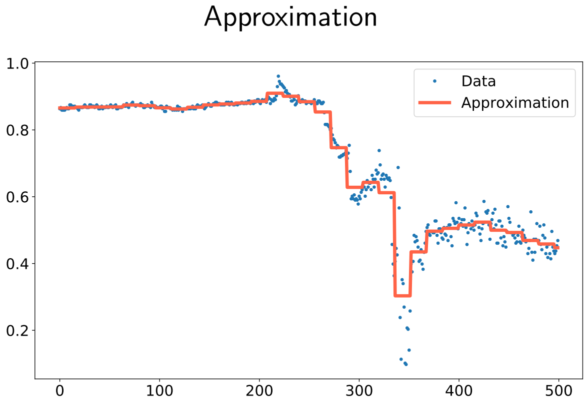
Approximation

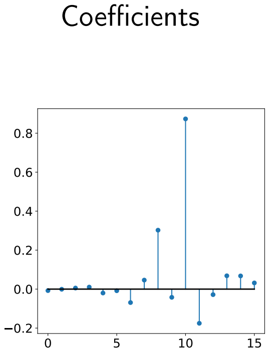


Coefficients

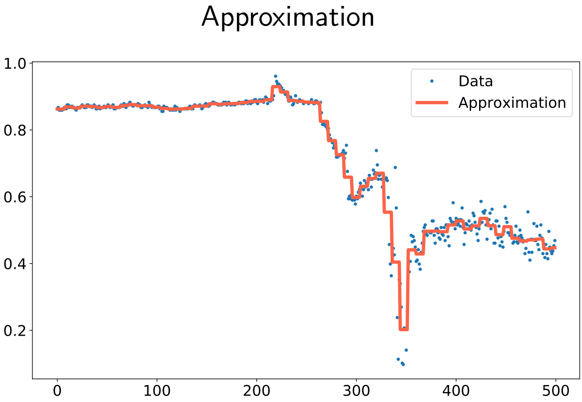


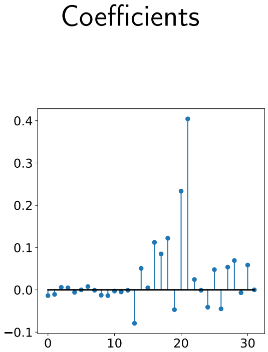
Father wavelet + 5 coarsest mother wavelets



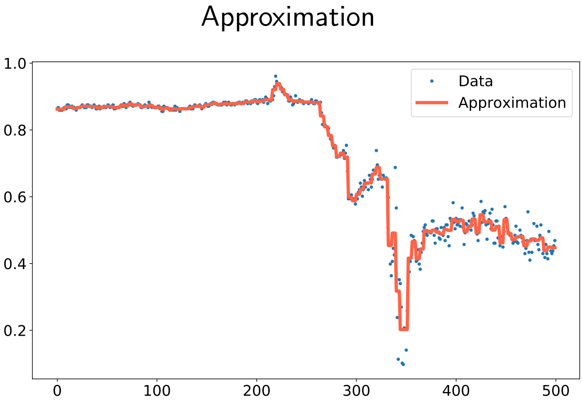


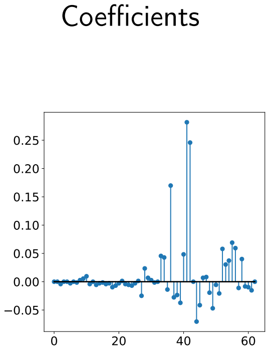
Father wavelet + 6 coarsest mother wavelets





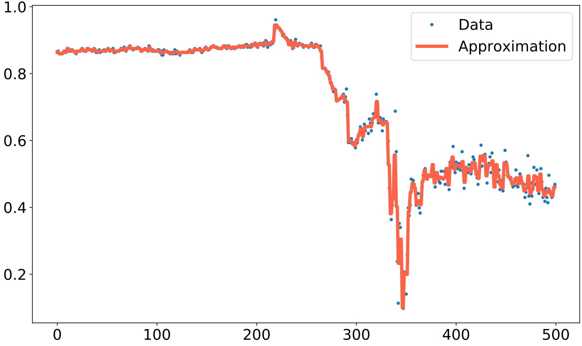
Father wavelet + 7 coarsest mother wavelets



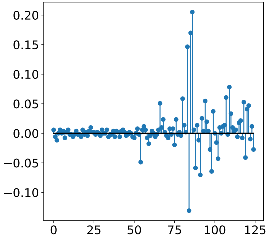


Father wavelet + 8 coarsest mother wavelets

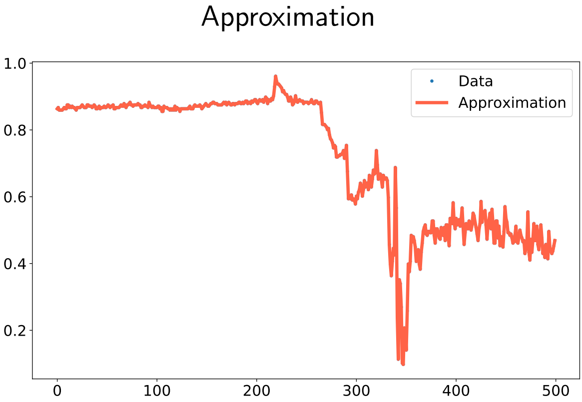
Approximation

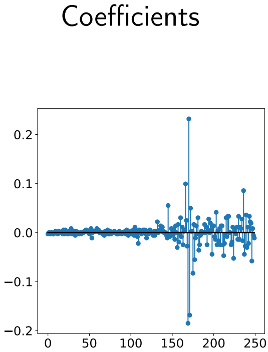


Coefficients



Father wavelet + 9 coarsest mother wavelets





Multiresolution decomposition

Let $N := 2^K$ for some K, a multiresolution decomposition of $\mathbb{R}^N$ is a sequence of nested subspaces $\mathcal{V}_K \subset \mathcal{V}_{K-1} \subset \ldots \subset \mathcal{V}_0$ satisfying:

- $\mathcal{V}_0 = \mathbb{R}^N$
- If $x \in \mathcal{V}_k$ then x shifted by 2^k is also in $\mathcal{V}_k$ (invariance to translations)
- Dilating $x \in \mathcal{V}_j$ yields vector in $\mathcal{V}_{j+1}$

Example

Subspace $\mathcal{V}_k$ contains vectors that are constant on segments of length 2^k

Satisfies conditions:

- $\mathcal{V}_0 = \mathbb{R}^N$
- If $x \in \mathcal{V}_k$ then x shifted by 2^k is also in $\mathcal{V}_k$ (invariance to translations)
- Dilating $x \in \mathcal{V}_j$ yields vector in $\mathcal{V}_{j+1}$

Spanned by shifts/dilations of Haar father wavelets

Problem: Basis vectors are not orthogonal (at all!)

Solution

Decompose the finer subspaces into a direct sum

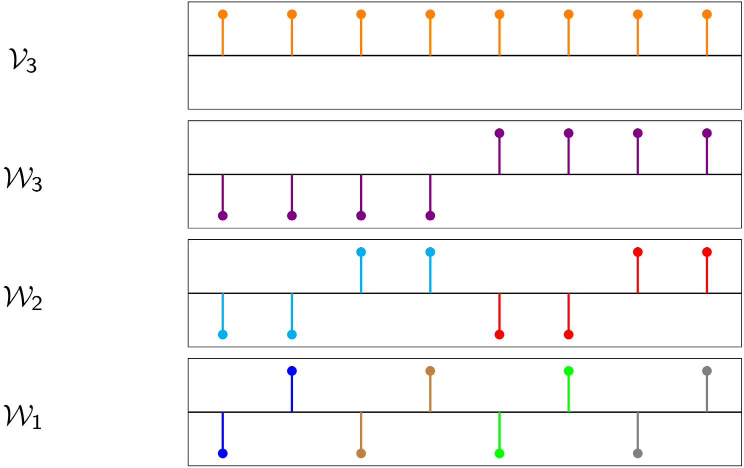
$$\mathcal{V}_k = \mathcal{V}_{k+1} \oplus \mathcal{W}_k, \qquad 0 \leq k \leq K-1,$$

$\mathcal{W}_k$ is the orthogonal complement of $\mathcal{V}_{k+1}$ in $\mathcal{V}_k$, so it captures finest resolution available at level k

We can then decompose $\mathbb{R}^N$ into different scales

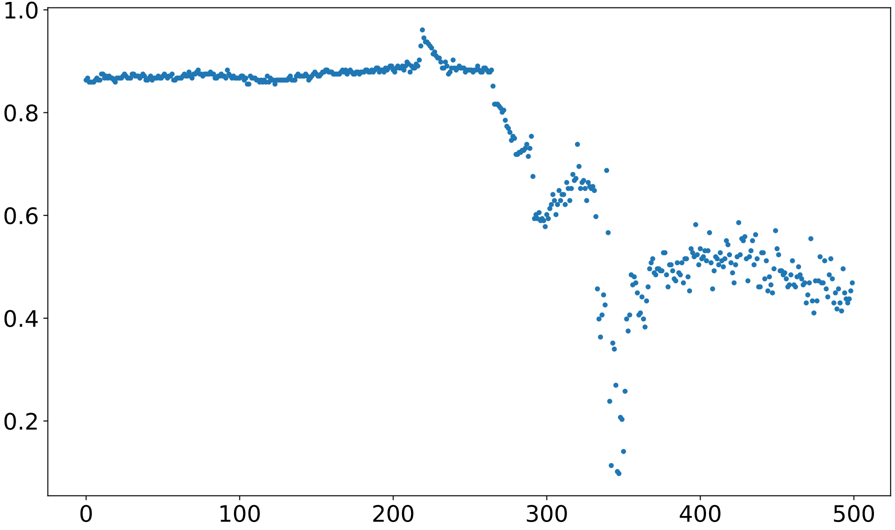
$$\begin{aligned}\mathbb{R}^N = \mathcal{V}_0 &= \mathcal{V}_1 \oplus \mathcal{W}_1 \\ &= \mathcal{V}_2 \oplus \mathcal{W}_2 \oplus \mathcal{W}_1 \\ &= \mathcal{V}_k \oplus \mathcal{W}_k \oplus \cdots \oplus \mathcal{W}_2 \oplus \mathcal{W}_1\end{aligned}$$

Haar multiresolution decomposition

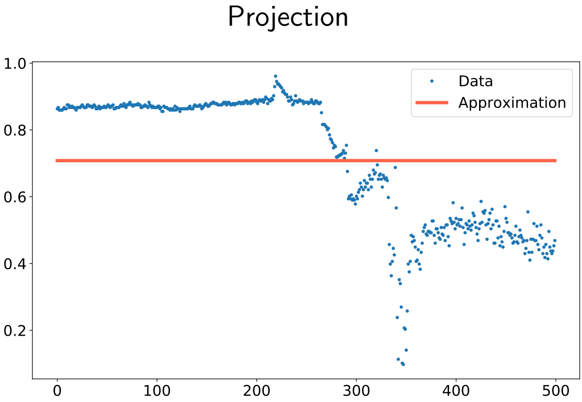


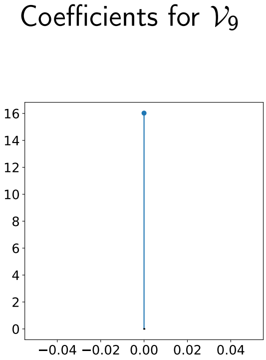
$\mathcal{P}_{\mathcal{V}_k}\, x$ is an approximation of x at scale 2^k

Vertical line (column 135)

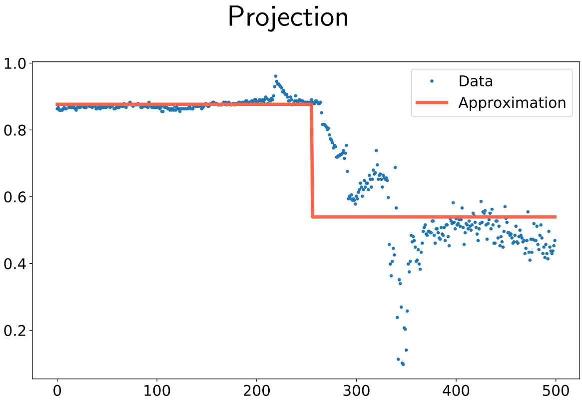


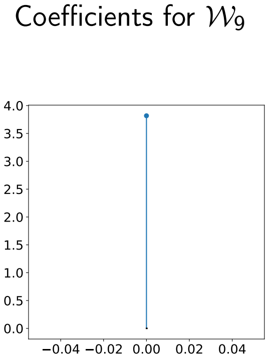
Projection onto $\mathcal{V}_9$



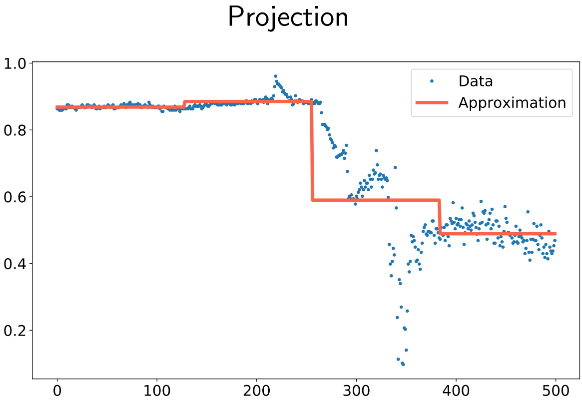


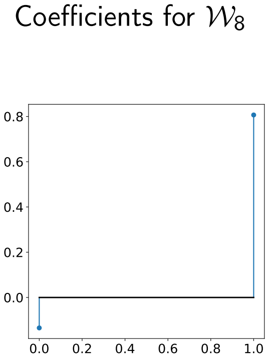
Projection onto $\mathcal{V}_8$



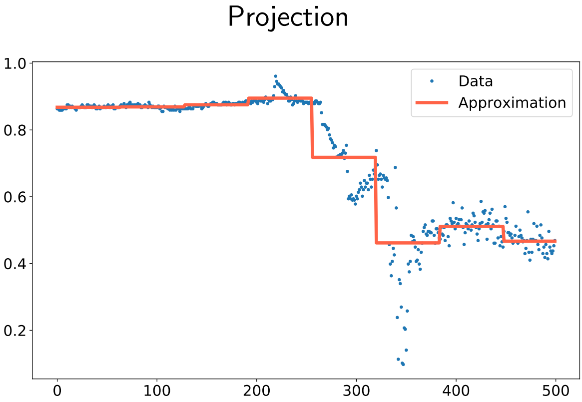


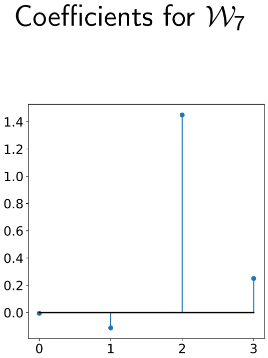
Projection onto $\mathcal{V}_7$





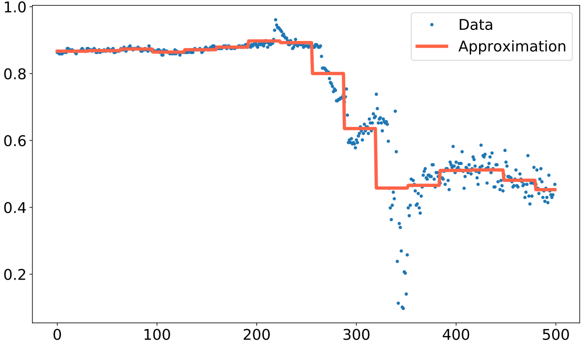
Projection onto $\mathcal{V}_6$



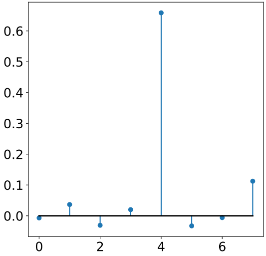


Projection onto $\mathcal{V}_5$

Projection

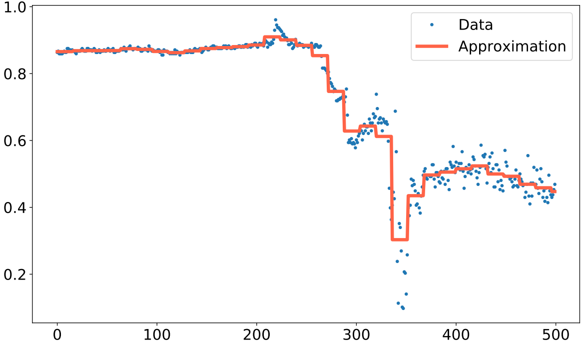


Coefficients for $\mathcal{W}_6$

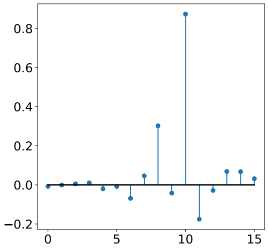


Projection onto $\mathcal{V}_4$

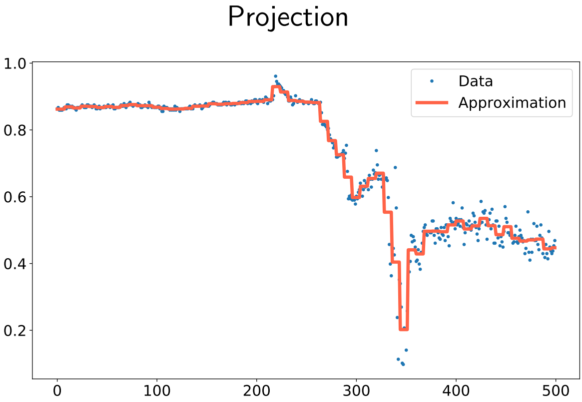
Projection

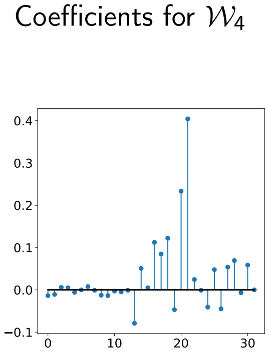


Coefficients for $\mathcal{W}_5$

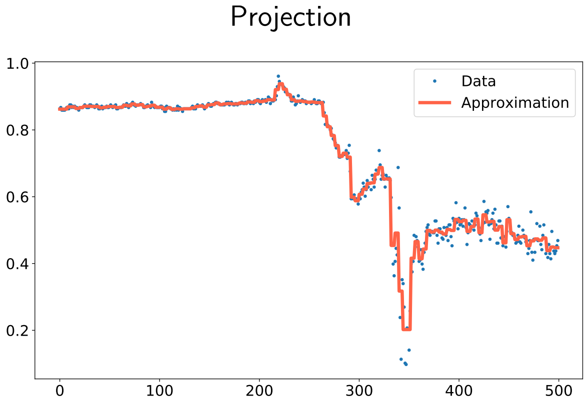


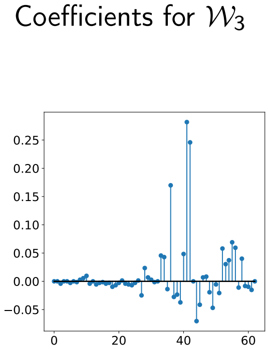
Projection onto $\mathcal{V}_3$



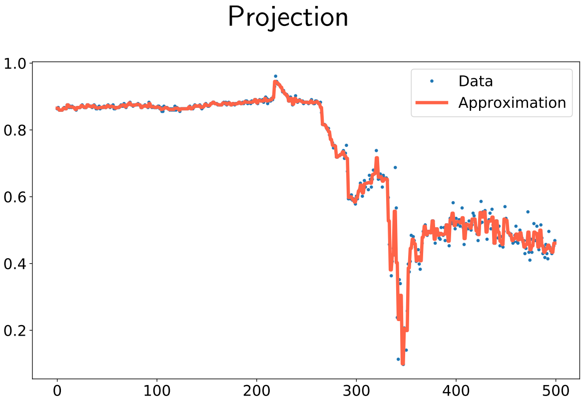


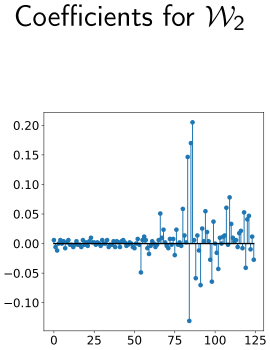
Projection onto $\mathcal{V}_2$



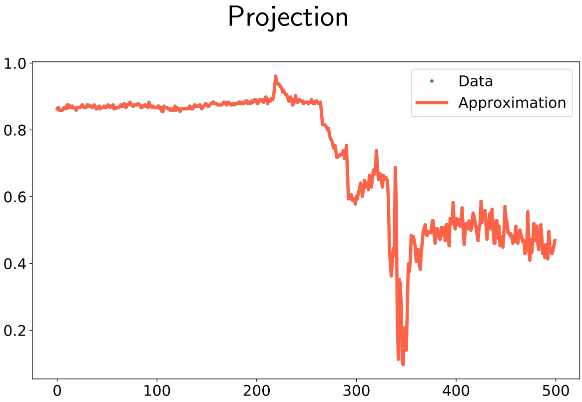


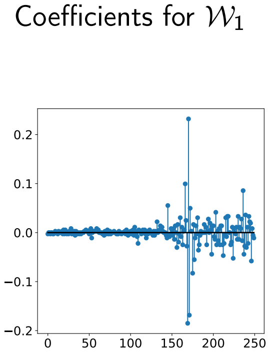
Projection onto $\mathcal{V}_1$



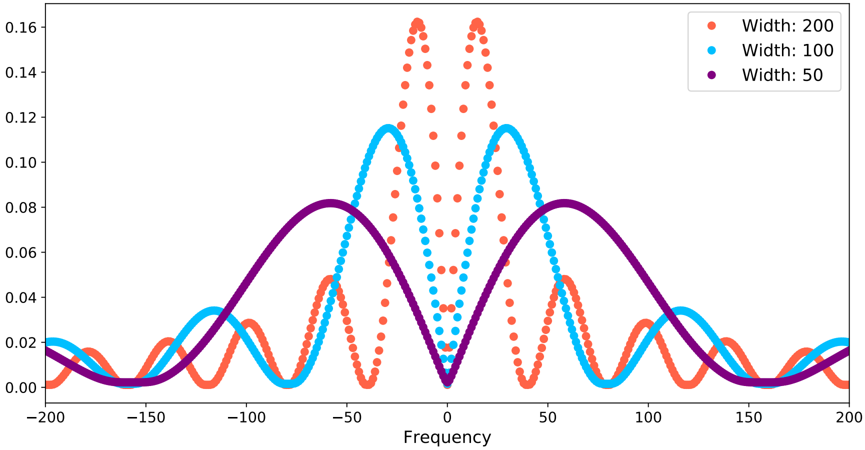


Projection onto $\mathcal{V}_0$



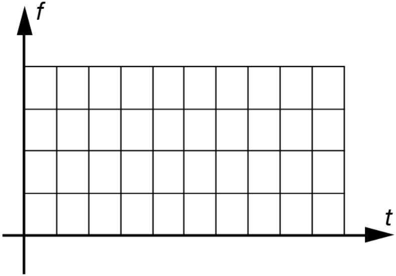


Haar mother wavelets in the frequency domain

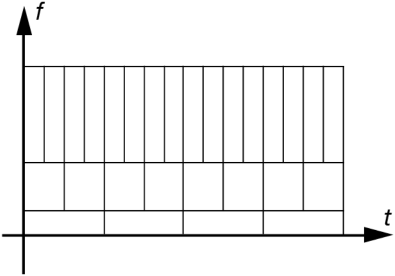


Time-frequency support of basis vectors

STFT



Wavelets



2D Wavelets

Extension to 2D by using outer products of 1D basis vectors

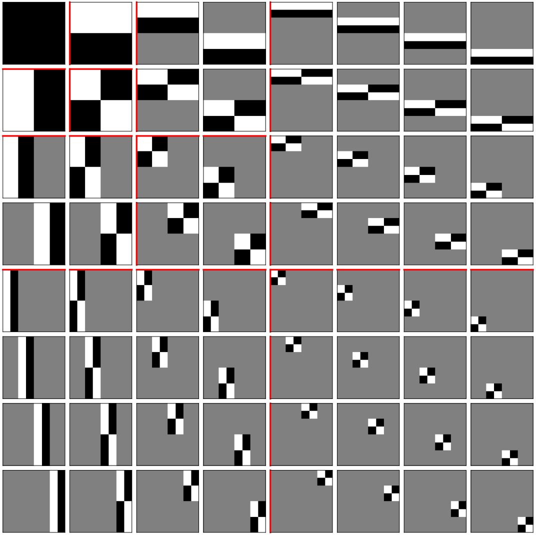
To build a 2D basis vector at scale (m_1, m_2) and shift (s_1, s_2) we set

$$v^{\mathrm{2D}}_{[s_1,s_2,m_1,m_2]} := v^{\mathrm{1D}}_{[s_1,m_1]} \left(v^{\mathrm{1D}}_{[s_2,m_2]} \right)^T,$$

where v^{1D} can refer to 1D father or mother wavelets

Nonseparable designs: steerable pyramid, curvelets, bandlets...

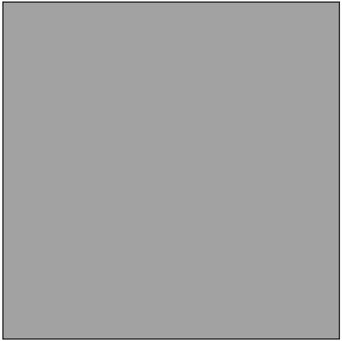
2D Haar wavelet basis vectors



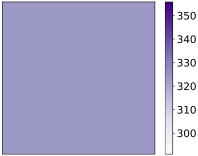
Image

2D Haar wavelet decomposition

Approximation

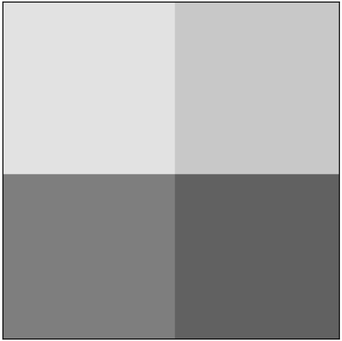


Coefficients

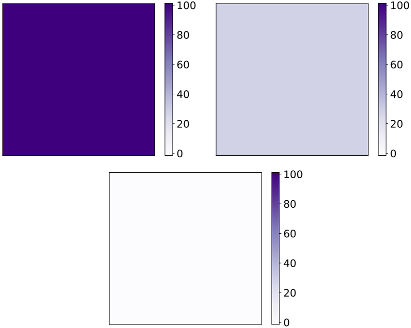


2D Haar wavelet decomposition

Approximation

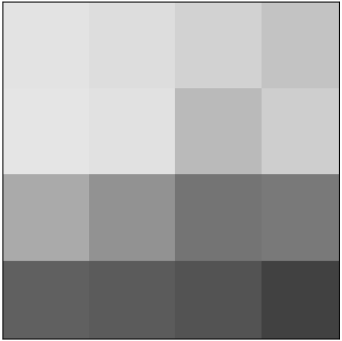


Coefficients

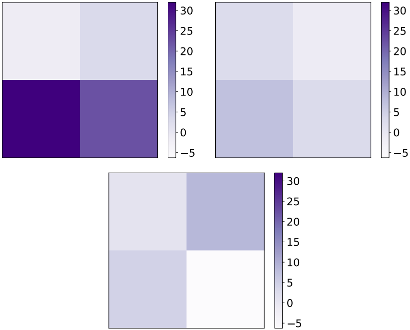


2D Haar wavelet decomposition

Approximation

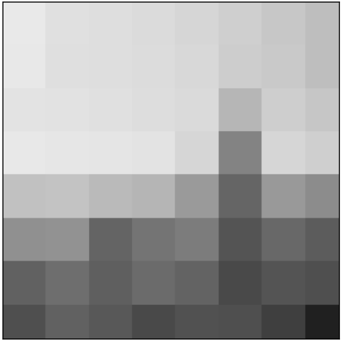


Coefficients

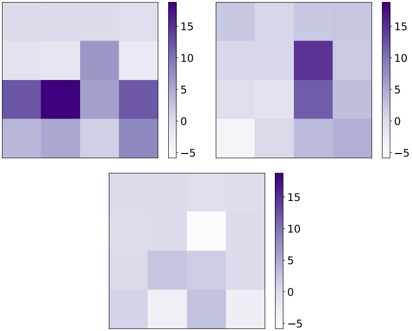


2D Haar wavelet decomposition

Approximation

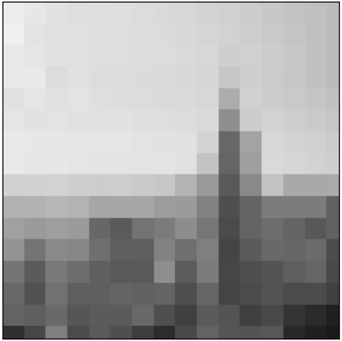


Coefficients

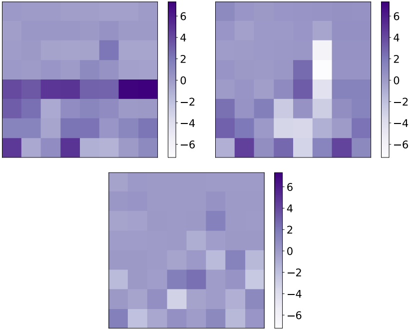


2D Haar wavelet decomposition

Approximation



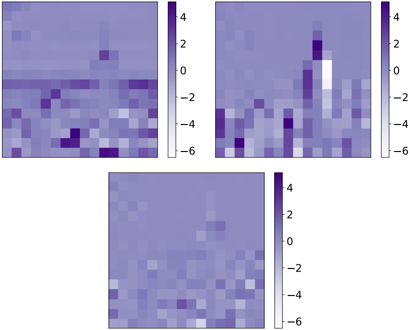
Coefficients



2D Haar wavelet decomposition

Approximation

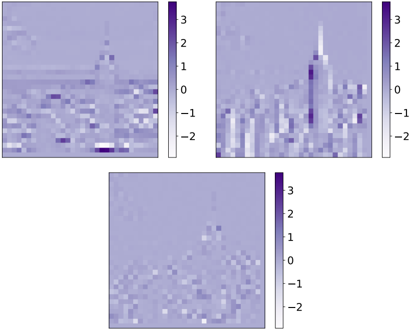
Coefficients



2D Haar wavelet decomposition

Approximation

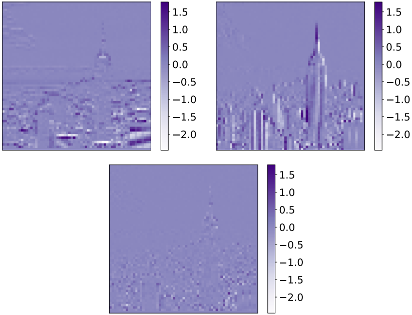
Coefficients



2D Haar wavelet decomposition

Approximation

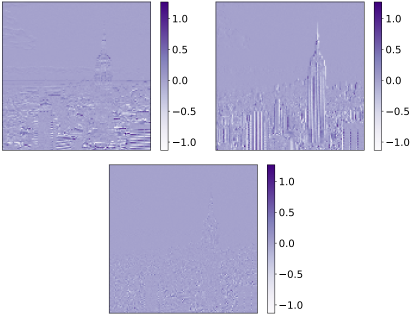
Coefficients



2D Haar wavelet decomposition

Approximation

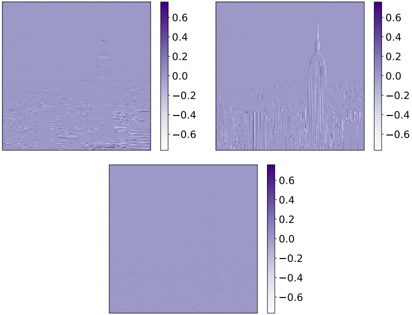
Coefficients



2D Haar wavelet decomposition

Approximation

Coefficients



What have we learned

Framework for multiresolution analysis based on wavelets

Implementation based on Haar wavelets

Extension to two dimensions