Windowing

DS-GA 1013 / MATH-GA 2824 Mathematical Tools for Data Science

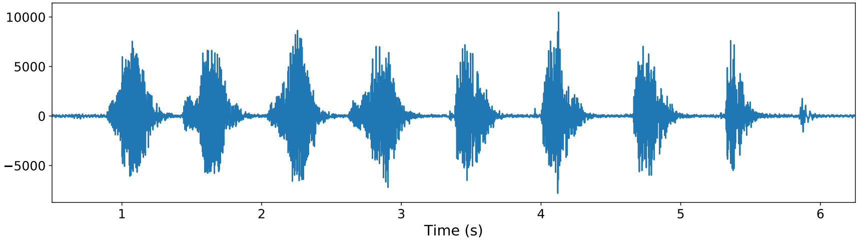
Carlos Fernandez-Granda

Prerequisites

Fourier series

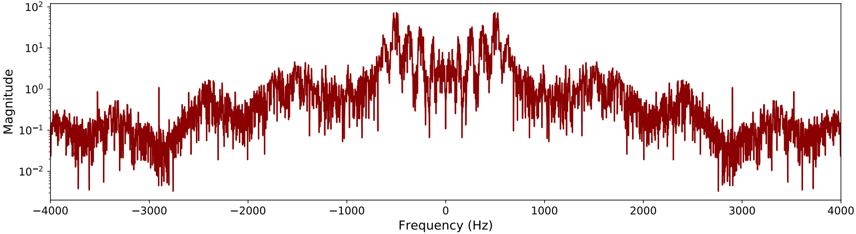
Discrete Fourier transform

Speech signal

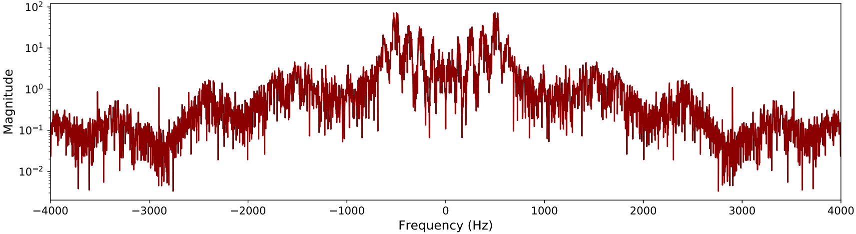


Challenge: Characterize how frequency components change *over time*

Fourier series

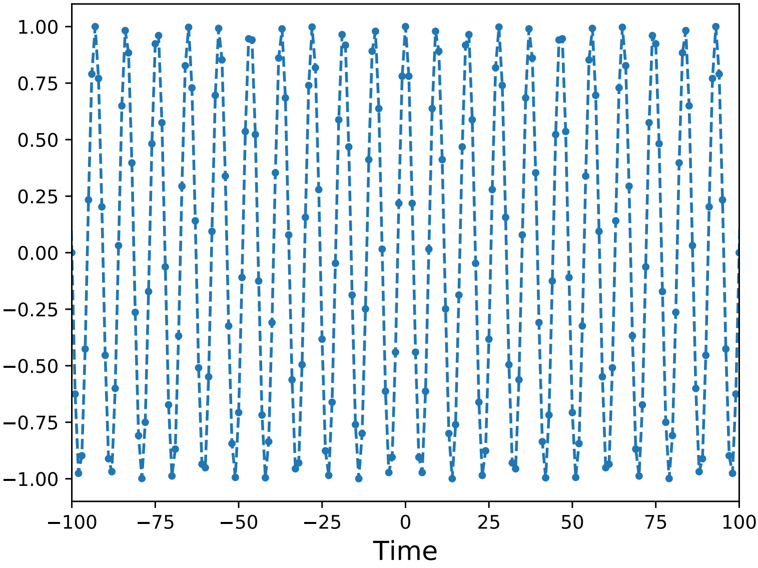


Fourier series



First segment or *window* signal, then compute Fourier series / DFT

Signal



1.00
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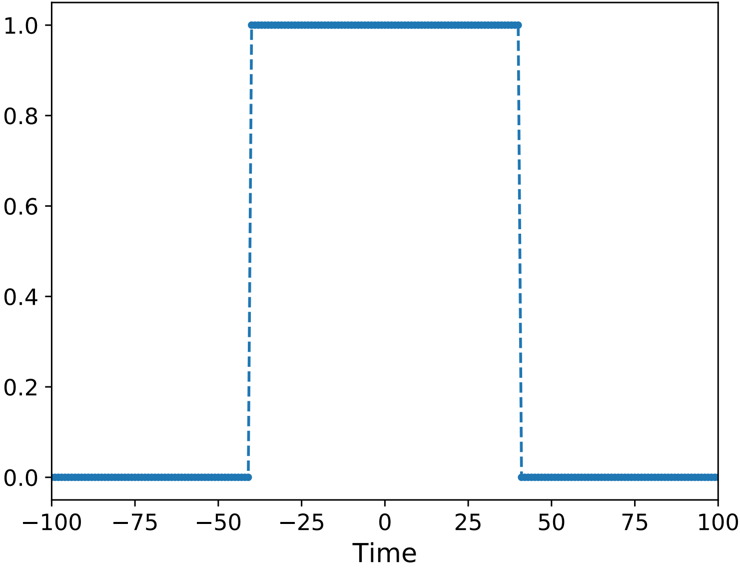
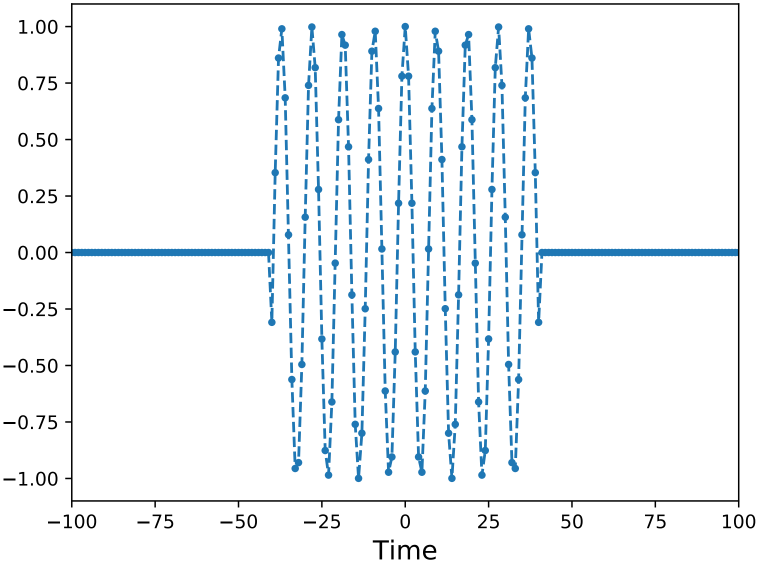
75
100
Time

Window



Windowed signal



Windowing a complex sinusoid

What happens if we window a sinusoid?

$$\psi_{k^*}[j] := \exp\left(\frac{i2\pi k^* j}{N}\right) \qquad \text{for } 0 \leq j \leq N-1$$

What is the DFT of $y := x\psi_{k^*}$?

Windowing a complex sinusoid

$$\hat{y}[k] := \sum_{j=1}^{N} x[j]\psi_{k^*}[j] \exp\left(-\frac{i2\pi kj}{N}\right)$$

Windowing a complex sinusoid

$$\hat{y}[k] := \sum_{j=1}^{N} x[j]\psi_{k^*}[j] \exp\left(-\frac{i2\pi kj}{N}\right)$$

$$= \sum_{j=1}^{N} x[j] \exp\left(\frac{i2\pi k^* j}{N}\right) \exp\left(-\frac{i2\pi kj}{N}\right)$$

Windowing a complex sinusoid

$$
\begin{aligned}
\hat{y}[k] &:= \sum_{j=1}^{N} x[j]\psi_{k^*}[j] \exp\left(-\frac{i2\pi kj}{N}\right) \\
&= \sum_{j=1}^{N} x[j] \exp\left(\frac{i2\pi k^* j}{N}\right) \exp\left(-\frac{i2\pi kj}{N}\right) \\
&= \sum_{j=1}^{N} x[j] \exp\left(\frac{-i2\pi (k-k^*) j}{N}\right)
\end{aligned}
$$

Windowing a complex sinusoid

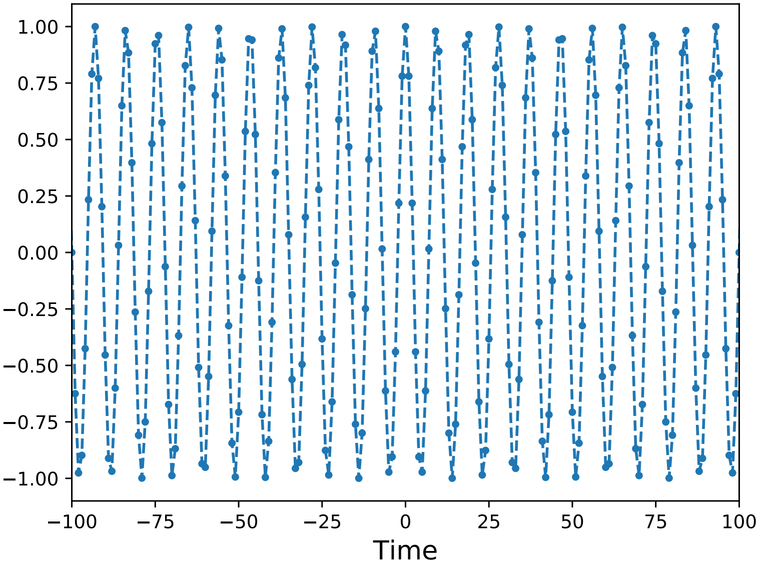
$$\begin{aligned}
\hat{y}\left[k\right] &:= \sum_{j=1}^{N} x[j]\psi_{k^*}[j] \exp\left(-\frac{i2\pi kj}{N}\right) \\
&= \sum_{j=1}^{N} x[j] \exp\left(\frac{i2\pi k^* j}{N}\right) \exp\left(-\frac{i2\pi kj}{N}\right) \\
&= \sum_{j=1}^{N} x[j] \exp\left(\frac{-i2\pi\left(k-k^*\right)j}{N}\right) \\
&= \hat{x}[(k-k^*)]
\end{aligned}$$

Windowing a real sinusoid

What happens if we window a real sinusoid?

$$s[j] := \cos\left(\frac{2\pi k^* j}{N}\right), \qquad 0 \leq j \leq N$$

Signal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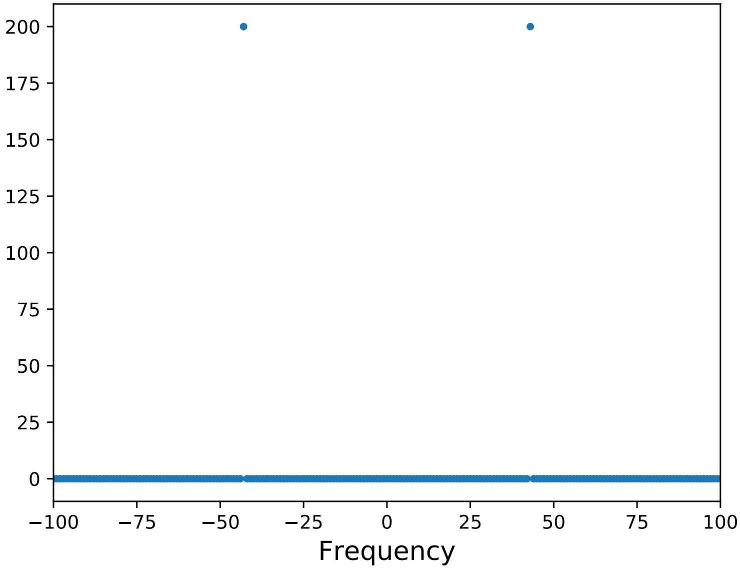
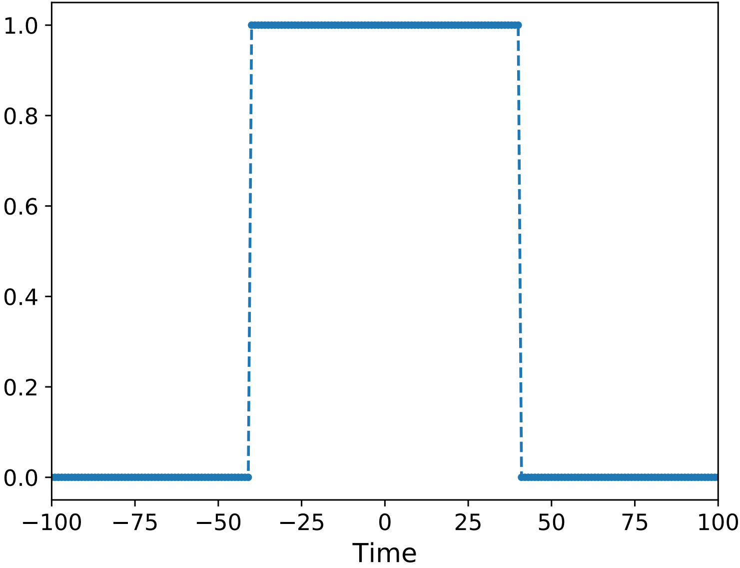
75
100
Time

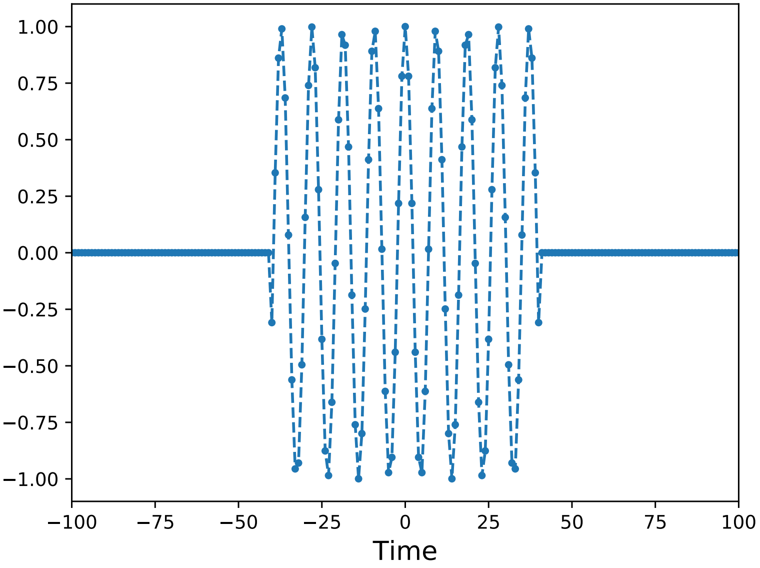
DFT of signal



Window



Windowed signal



Windowing a real sinusoid

What happens if we window a real sinusoid?

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What is the DFT of $y := xs$?

$$xs = x\left(\frac{\psi_{k^*} + \psi_{-k^*}}{2}\right)$$

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$$\widehat{xs} = \frac{\widehat{x\psi_{k^*}} + \widehat{x\psi_{-k^*}}}{2}$$

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What is the DFT of $y := xs$?

$$xs = x\left(\frac{\psi_{k^*} + \psi_{-k^*}}{2}\right)$$

$$\begin{aligned}\widehat{xs} &= \frac{\widehat{x\psi_{k^*}} + \widehat{x\psi_{-k^*}}}{2} \\ &= \frac{\hat{x}[(k - k^*)] + \hat{x}[k + k^*]}{2}, \qquad 0 \le k \le N\end{aligned}$$

Rectangular window

Rectangular window $\vec{\pi} \in \mathbb{C}^N$ with width $2w$:

$$\vec{\pi}[j] := \begin{cases} 1 & \text{if } |j| \leq w, \\ 0 & \text{otherwise} \end{cases}$$

DFT of rectangular window

$$\hat{\pi}[0] = \sum_{j=-N/2+1}^{N/2} \vec{\pi}[j]$$

$$= \sum_{j=-w}^{w} 1 = 2w + 1$$

DFT of rectangular window

$$\hat{\pi}[k] = \sum_{j=-N/2+1}^{N/2} \vec{\pi}[j] \exp\left(-\frac{i2\pi kj}{N}\right)$$

DFT of rectangular window

$$\begin{aligned}\hat{\pi}[k] &= \sum_{j=-N/2+1}^{N/2} \vec{\pi}[j] \exp\left(-\frac{i2\pi kj}{N}\right) \\ &= \sum_{j=-w}^{w} \exp\left(-\frac{i2\pi k}{N}\right)^{j}\end{aligned}$$

DFT of rectangular window

$$
\begin{aligned}
\hat{\pi}[k] &= \sum_{j=-N/2+1}^{N/2} \vec{\pi}[j] \exp\left(-\frac{i2\pi kj}{N}\right) \\
&= \sum_{j=-w}^{w} \exp\left(-\frac{i2\pi k}{N}\right)^{j} \\
&= \frac{\exp\left(\frac{i2\pi kw}{N}\right) - \exp\left(-\frac{i2\pi k(w+1)}{N}\right)}{1 - \exp\left(-\frac{i2\pi k}{N}\right)}
\end{aligned}
$$

DFT of rectangular window

$$\begin{aligned}
\hat{\pi}[k] &= \sum_{j=-N/2+1}^{N/2} \vec{\pi}[j] \exp\left(-\frac{i2\pi kj}{N}\right) \\
&= \sum_{j=-w}^{w} \exp\left(-\frac{i2\pi k}{N}\right)^{j} \\
&= \frac{\exp\left(\frac{i2\pi kw}{N}\right) - \exp\left(-\frac{i2\pi k(w+1)}{N}\right)}{1 - \exp\left(-\frac{i2\pi k}{N}\right)} \\
&= \frac{\exp\left(-\frac{i2\pi k}{2N}\right) 2i \sin\left(\frac{2\pi k(w+1/2)}{N}\right)}{\exp\left(-\frac{i2\pi k}{2N}\right) 2i \sin\left(\frac{\pi k}{N}\right)}
\end{aligned}$$

DFT of rectangular window

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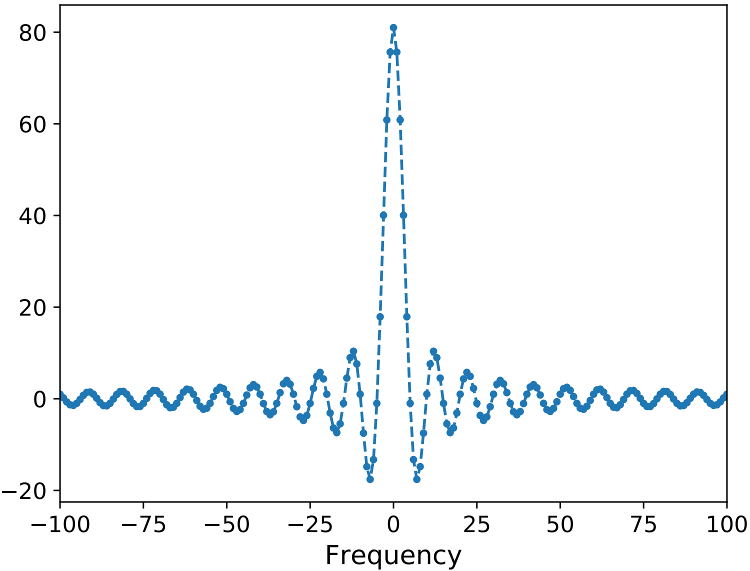
\hat{\pi}[k] &= \sum_{j=-N/2+1}^{N/2} \vec{\pi}[j] \exp\left(-\frac{i2\pi kj}{N}\right) \\
&= \sum_{j=-w}^{w} \exp\left(-\frac{i2\pi k}{N}\right)^{j} \\
&= \frac{\exp\left(\frac{i2\pi kw}{N}\right) - \exp\left(-\frac{i2\pi k(w+1)}{N}\right)}{1 - \exp\left(-\frac{i2\pi k}{N}\right)} \\
&= \frac{\exp\left(-\frac{i2\pi k}{2N}\right) 2i \sin\left(\frac{2\pi k(w+1/2)}{N}\right)}{\exp\left(-\frac{i2\pi k}{2N}\right) 2i \sin\left(\frac{\pi k}{N}\right)} \\
&= \frac{\sin\left(\frac{2\pi k(w+1/2)}{N}\right)}{\sin\left(\frac{\pi k}{N}\right)}
\end{aligned}$$

DFT of rectangular window



Windowing a real sinusoid

What happens if we window a real sinusoid?

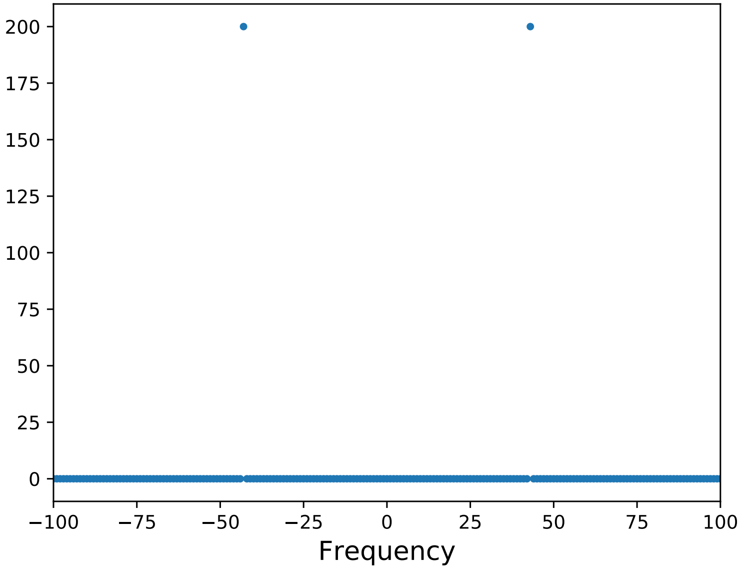
$$s[j] := \cos\left(\frac{2\pi k^* j}{N}\right), \qquad 0 \le j \le N$$

What is the DFT of $y := xs$?

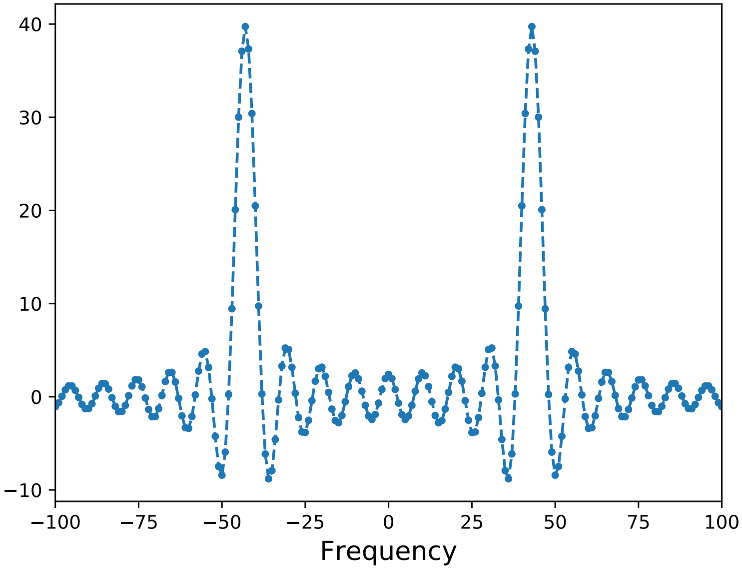
$$xs = x\left(\frac{\psi_{k^*} + \psi_{-k^*}}{2}\right)$$

$$\begin{aligned}\widehat{xs} &= \frac{\widehat{x\psi_{k^*}} + \widehat{x\psi_{-k^*}}}{2} \\ &= \frac{\hat{x}[(k-k^*)] + \hat{x}[-(k-k^*)]}{2}, \qquad 0 \le k \le N\end{aligned}$$

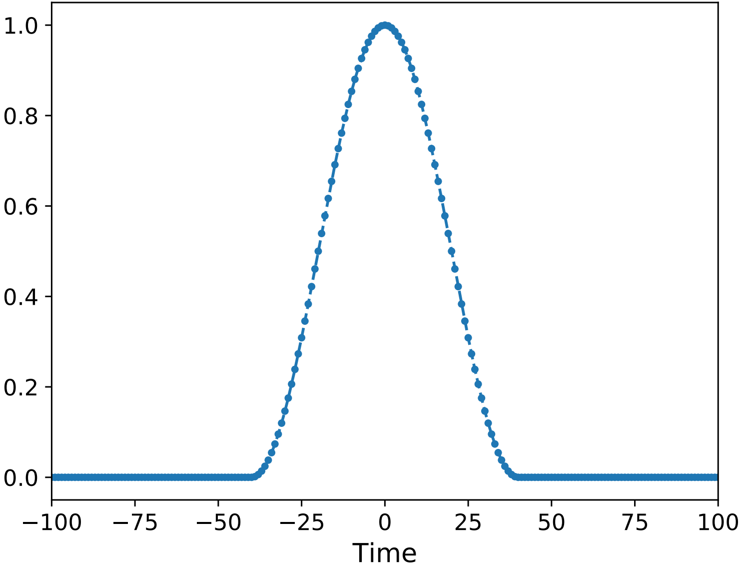
DFT of signal



DFT of windowed signal



Hann window

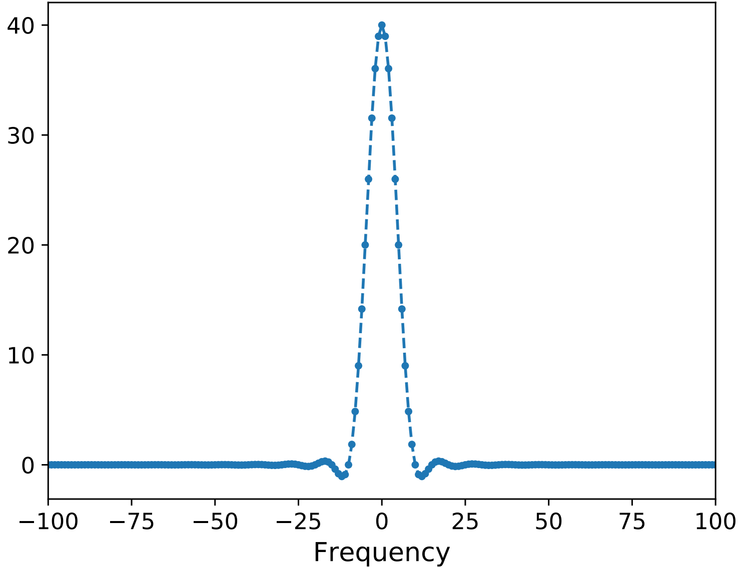


Hann window

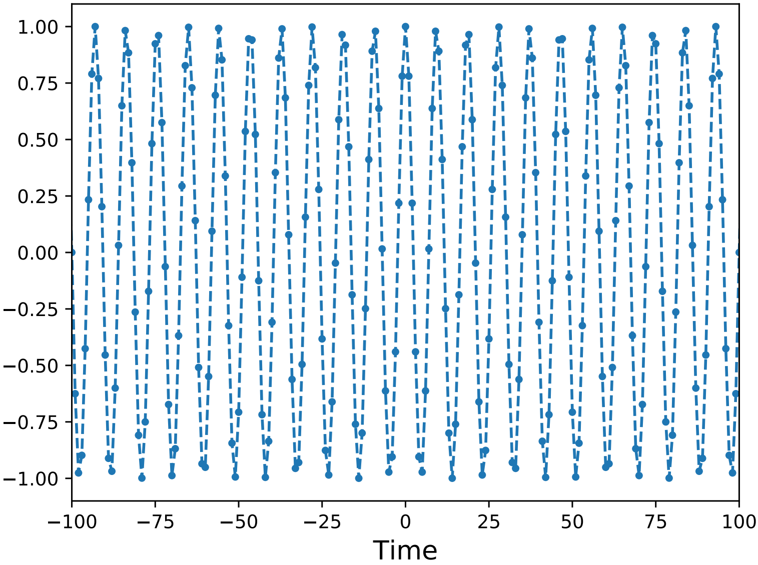
The Hann window $h \in \mathbb{C}^N$ of width $2w$ equals

$$h[j] := \begin{cases} \frac{1}{2}\left(1 + \cos\left(\frac{\pi j}{w}\right)\right) & \text{if } |j| \leq w, \\ 0 & \text{otherwise} \end{cases}$$

DFT of Hann window



Signal



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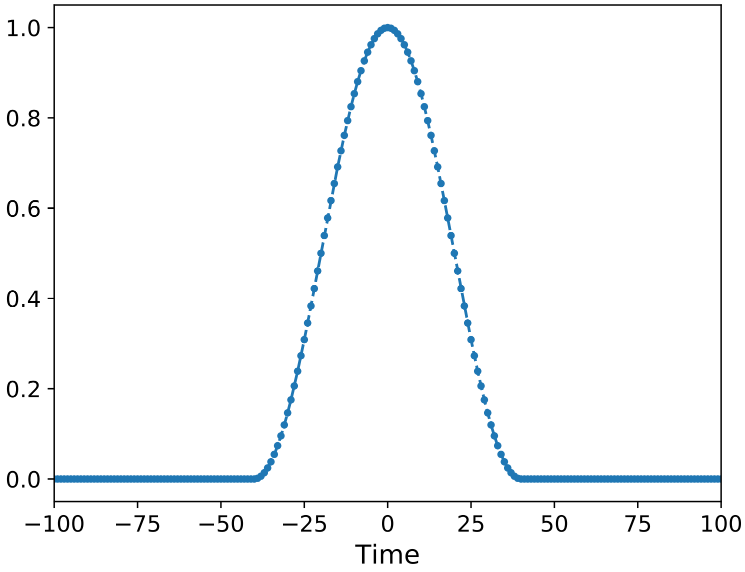
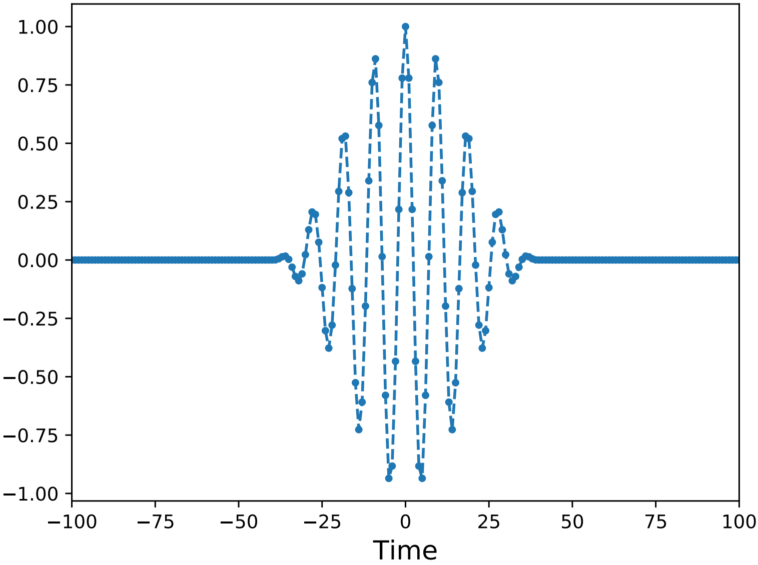
25
50
75
100
Time

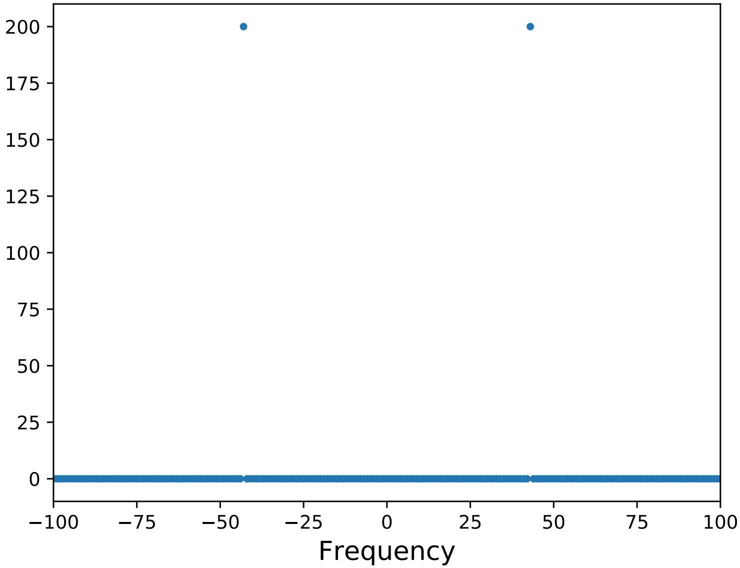
Hann window



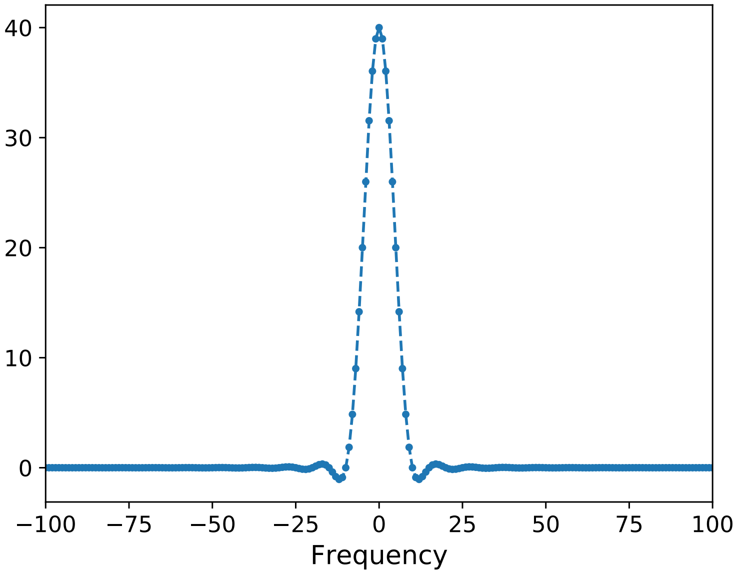
Windowed signal



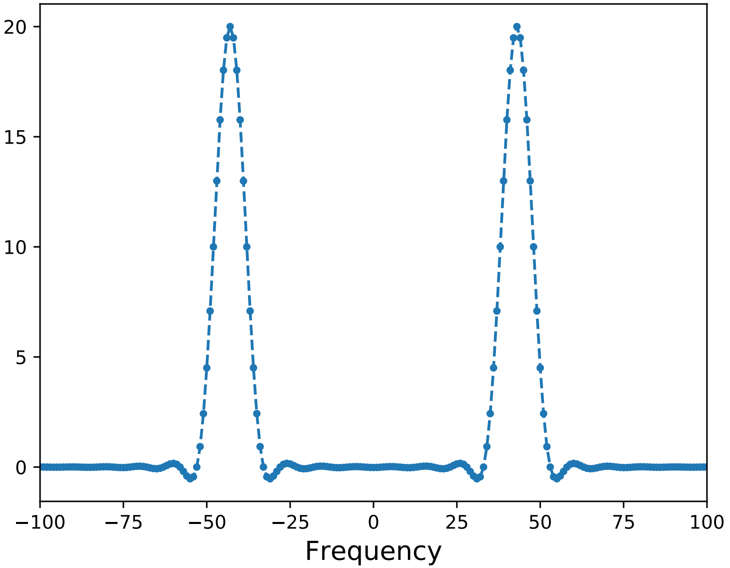
DFT of signal



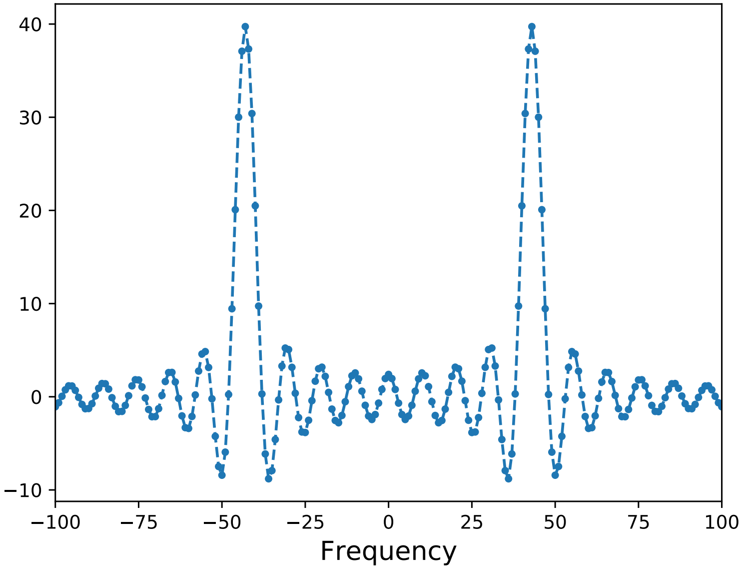
DFT of Hann window



DFT of windowed signal



DFT of windowed signal (rectangular window)



Time-frequency resolution

Time resolution governed by width of window

Can we just make the window arbitrarily narrow?

Compressing in time dilates in frequency and vice versa

$x \in \mathcal{L}_2\left[-T/2, T/2\right]$ is nonzero in a band of width $2w$ around zero

Let y be such that

$$y(t) = x(\alpha t), \quad \text{for all } t \in \left[-T/2, T/2\right],$$

for some positive real number α such that $w/\alpha < T$

The Fourier series coefficients of y equal

$$\hat{y}\left[k\right] = \frac{1}{\alpha}\left\langle x, \phi_{k/\alpha}\right\rangle$$

Proof

$$\hat{y}[k] = \int_{t=-T/2}^{T/2} y(t) \exp\left(-\frac{i2\pi kt}{T}\right) \, \mathrm{d}t$$

Proof

$$\hat{y}[k] = \int_{t=-T/2}^{T/2} y(t) \exp\left(-\frac{i2\pi kt}{T}\right) \, \mathrm{d}t$$
$$= \int_{t=-w/\alpha}^{w/\alpha} x(\alpha t) \exp\left(-\frac{i2\pi kt}{T}\right) \, \mathrm{d}t$$

Proof

$$\begin{aligned}
\hat{y}\left[k\right] &= \int_{t=-T/2}^{T/2} y(t) \exp\left(-\frac{i2\pi kt}{T}\right)\, \mathrm{d}t \\
&= \int_{t=-w/\alpha}^{w/\alpha} x(\alpha t) \exp\left(-\frac{i2\pi kt}{T}\right)\, \mathrm{d}t \\
&= \frac{1}{\alpha}\int_{\tau=-w}^{w} x(\tau) \exp\left(-\frac{i2\pi k\tau}{\alpha T}\right)\, \mathrm{d}\tau
\end{aligned}$$

Proof

$$
\begin{aligned}
\hat{y}[k] &= \int_{t=-T/2}^{T/2} y(t) \exp\left(-\frac{i2\pi kt}{T}\right) \mathrm{d}t \\
&= \int_{t=-w/\alpha}^{w/\alpha} x(\alpha t) \exp\left(-\frac{i2\pi kt}{T}\right) \mathrm{d}t \\
&= \frac{1}{\alpha} \int_{\tau=-w}^{w} x(\tau) \exp\left(-\frac{i2\pi k\tau}{\alpha T}\right) \mathrm{d}\tau \\
&= \frac{1}{\alpha} \int_{\tau=-T/2}^{T/2} x(\tau) \exp\left(-\frac{i2\pi k\tau}{\alpha T}\right) \mathrm{d}\tau
\end{aligned}
$$

Proof

$$
\begin{aligned}
\hat{y}[k] &= \int_{t=-T/2}^{T/2} y(t) \exp\left(-\frac{i2\pi kt}{T}\right) \mathrm{d}t \\
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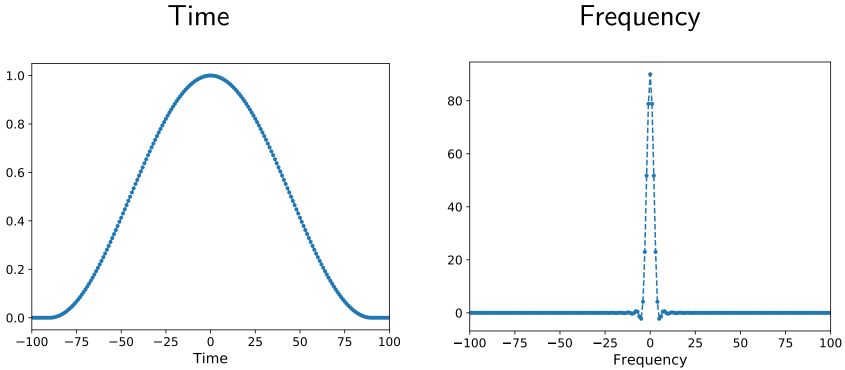
&= \frac{1}{\alpha} \left\langle x, \phi_{k/\alpha} \right\rangle
\end{aligned}
$$

$w = 90$



Time
1.0
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Time
Frequency
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−100
−75
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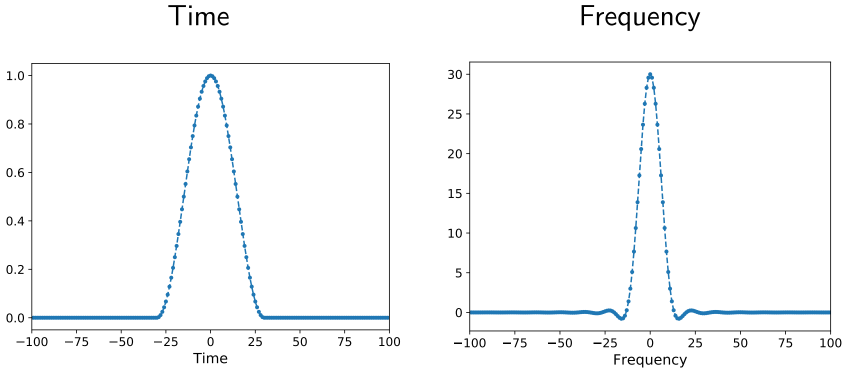
−25
0
25
50
75
100
Frequency

$w = 30$



Time
Frequency
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0.8
0.6
0.4
0.2
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−75
−50
−25
0
25
50
75
100
Time
30
25
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−100
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−25
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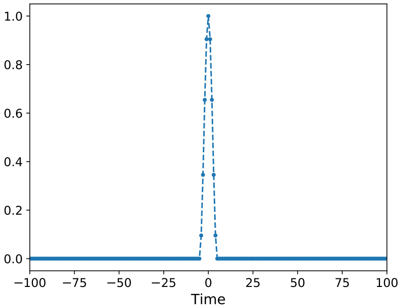
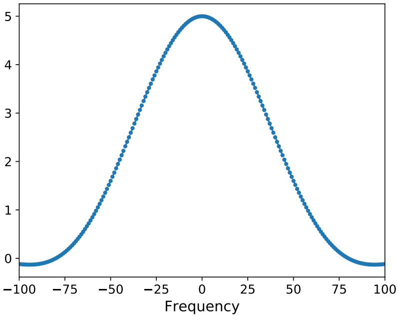
25
50
75
100
Frequency

$w = 5$

Time



Frequency



Time-frequency resolution

Fundamental trade-off

Uncertainty principle: cannot resolve in time and frequency simultaneously

What have we learned

Effect of temporal windowing in the frequency domain

Trade-off in time-frequency resolution