Windowing

DS-GA 1013 / MATH-GA 2824 Mathematical Tools for Data Science

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Prerequisites

Fourier series

Discrete Fourier transform
Challenge: Characterize how frequency components change over time
Fourier series

First segment or window signal, then compute Fourier series / DFT
First segment or *window* signal, then compute Fourier series / DFT
Signal
Windowed signal
Windowing a complex sinusoid

What happens if we window a sinusoid?

\[ \psi_{k^*}[j] := \exp \left( \frac{i2\pi k^* j}{N} \right) \quad \text{for } 0 \leq j \leq N - 1 \]

What is the DFT of \( y := x \psi_{k^*} \)?
Windowing a complex sinusoid

\[ \hat{y} [k] := \sum_{j=1}^{N} x[j] \psi_k^* [j] \exp \left( - \frac{i2\pi kj}{N} \right) \]
Windowing a complex sinusoid

\[ \hat{y}[k] := \sum_{j=1}^{N} x[j] \psi_{k^*}[j] \exp \left( -\frac{i2\pi kj}{N} \right) \]

\[ = \sum_{j=1}^{N} x[j] \exp \left( \frac{i2\pi k^*j}{N} \right) \exp \left( -\frac{i2\pi kj}{N} \right) \]
Windowing a complex sinusoid

\[ \hat{y}[k] := \sum_{j=1}^{N} x[j] \psi_{k^*}[j] \exp \left( -\frac{i2\pi kj}{N} \right) \]

\[ = \sum_{j=1}^{N} x[j] \exp \left( \frac{i2\pi k^*j}{N} \right) \exp \left( -\frac{i2\pi kj}{N} \right) \]

\[ = \sum_{j=1}^{N} x[j] \exp \left( -\frac{i2\pi (k - k^*) j}{N} \right) \]
Windowing a complex sinusoid

\[ \hat{y}[k] := \sum_{j=1}^{N} x[j] \psi_{k^*}[j] \exp\left( -\frac{i2\pi kj}{N} \right) \]

\[ = \sum_{j=1}^{N} x[j] \exp\left( \frac{i2\pi k^*j}{N} \right) \exp\left( -\frac{i2\pi kj}{N} \right) \]

\[ = \sum_{j=1}^{N} x[j] \exp\left( -\frac{i2\pi (k - k^*) j}{N} \right) \]

\[ = \hat{x}[ (k - k^*) ] \]
Windowing a real sinusoid

What happens if we window a real sinusoid?

\[ s[j] := \cos \left( \frac{2\pi k^* j}{N} \right), \quad 0 \leq j \leq N \]
Signal
DFT of signal
Window

The graph shows a step function with a jump from 0 to 1 at time 0.2 and back to 0 at time 0.5. The x-axis represents time, ranging from -100 to 100, and the y-axis represents the function value, ranging from 0 to 1.
Windowed signal
Windowing a real sinusoid

What happens if we window a real sinusoid?

\[ s[j] := \cos \left( \frac{2\pi k^* j}{N} \right) , \quad 0 \leq j \leq N \]

What is the DFT of \( y := xs \)?
Windowing a real sinusoid

What happens if we window a real sinusoid?

\[ s[j] := \cos\left(\frac{2\pi k^* j}{N}\right), \quad 0 \leq j \leq N \]

What is the DFT of \( y := xs \)?

\[ xs = x \left( \frac{\psi_{k^*} + \psi_{-k^*}}{2} \right) \]
Windowing a real sinusoid

What happens if we window a real sinusoid?

\[ s[j] := \cos \left( \frac{2\pi k^* j}{N} \right), \quad 0 \leq j \leq N \]

What is the DFT of \( y := xs \)?

\[ \hat{xs} = x \left( \frac{\psi_{k^*} + \psi_{-k^*}}{2} \right) \]

\[ \hat{xs} = \frac{x \psi_{k^*} + x \psi_{-k^*}}{2} \]
Windowing a real sinusoid

What happens if we window a real sinusoid?

\[ s[j] := \cos \left( \frac{2\pi k^* j}{N} \right), \quad 0 \leq j \leq N \]

What is the DFT of \( y := xs \)?

\[
xs = x \left( \frac{\psi_{k^*} + \psi_{-k^*}}{2} \right)
\]

\[
\hat{xs} = \frac{\hat{x}\psi_{k^*} + \hat{x}\psi_{-k^*}}{2} = \frac{\hat{x}[(k - k^*)] + \hat{x}[k + k^*]}{2}, \quad 0 \leq k \leq N
\]
Rectangular window \( \tilde{\pi} \in \mathbb{C}^N \) with width \( 2w \):

\[
\tilde{\pi}[j] := \begin{cases} 
1 & \text{if } |j| \leq w, \\
0 & \text{otherwise}
\end{cases}
\]
DFT of rectangular window

\[ \hat{\pi}[0] = \sum_{j=-N/2+1}^{N/2} \hat{\pi}[j] \]

\[ = \sum_{j=-w}^{w} 1 = 2w + 1 \]
DFT of rectangular window

\[ \hat{\pi}[k] = \sum_{j=-N/2+1}^{N/2} \pi[j] \exp \left( -\frac{i2\pi kj}{N} \right) \]
DFT of rectangular window

\[
\hat{\pi} [k] = \sum_{j=-N/2+1}^{N/2} \pi [j] \exp \left( -\frac{i2\pi k j}{N} \right)
\]

\[
= \sum_{j=-w}^{w} \exp \left( -\frac{i2\pi k j}{N} \right)
\]
DFT of rectangular window

\[
\hat{\pi}[k] = \sum_{j=-N/2+1}^{N/2} \hat{\pi}[j] \exp \left( -\frac{i2\pi kj}{N} \right)
\]

\[
= \sum_{j=-w}^{w} \exp \left( -\frac{i2\pi k}{N} \right)^j
\]

\[
= \exp \left( \frac{i2\pi kw}{N} \right) - \exp \left( -\frac{i2\pi k(w+1)}{N} \right)
\]

\[
= \frac{\exp \left( -\frac{i2\pi k}{N} \right) - 1}{1 - \exp \left( -\frac{i2\pi k}{N} \right)}
\]
DFT of rectangular window

\[ \hat{\pi} [k] = \sum_{j=-N/2+1}^{N/2} \hat{\pi} [j] \exp \left( -\frac{i2\pi kj}{N} \right) \]

\[ = \sum_{j=-w}^{w} \exp \left( -\frac{i2\pi k}{N} \right)^j \]

\[ = \frac{\exp \left( \frac{i2\pi kw}{N} \right) - \exp \left( -\frac{i2\pi k(w+1)}{N} \right)}{1 - \exp \left( -\frac{i2\pi k}{N} \right)} \]

\[ = \frac{\exp \left( -\frac{i2\pi k}{2N} \right) 2i \sin \left( \frac{2\pi k(w+1/2)}{N} \right)}{\exp \left( -\frac{i2\pi k}{2N} \right) 2i \sin \left( \frac{\pi k}{N} \right)} \]
DFT of rectangular window

\[ \hat{\pi} [k] = \sum_{j=-N/2+1}^{N/2} \hat{\pi} [j] \exp \left( -\frac{i 2\pi k j}{N} \right) \]

\[ = \sum_{j=-w}^{w} \exp \left( -\frac{i 2\pi k}{N} \right)^j \]

\[ = \exp \left( \frac{i 2\pi kw}{N} \right) - \exp \left( -\frac{i 2\pi k(w+1)}{N} \right) \]

\[ = \frac{\exp \left( -\frac{i 2\pi k}{2N} \right) 2i \sin \left( \frac{2\pi k(w+1/2)}{N} \right)}{1 - \exp \left( -\frac{i 2\pi k}{N} \right)} \]

\[ = \frac{\exp \left( -\frac{i 2\pi k}{2N} \right) 2i \sin \left( \frac{\pi k}{N} \right)}{\sin \left( \frac{\pi k}{N} \right)} \]
DFT of rectangular window
Windowing a real sinusoid

What happens if we window a real sinusoid?

\[ s[j] := \cos \left( \frac{2\pi k^* j}{N} \right), \quad 0 \leq j \leq N \]

What is the DFT of \( y := xs \)?

\[ xs = x \left( \frac{\psi_{k^*} + \psi_{-k^*}}{2} \right) \]

\[ \hat{x} = \frac{x\psi_{k^*} + x\psi_{-k^*}}{2} \]

\[ = \frac{\hat{x}[(k - k^*)] + \hat{x}[-(k - k^*)]}{2}, \quad 0 \leq k \leq N \]
DFT of signal
DFT of windowed signal
Hann window

![Graph of a Hann window function]

- Time axis from 0 to 100
- Value axis from -100 to 100
The Hann window \( h \in \mathbb{C}^N \) of width \( 2w \) equals

\[
    h[j] := \begin{cases} 
        \frac{1}{2} \left( 1 + \cos \left( \frac{\pi j}{w} \right) \right) & \text{if } |j| \leq w, \\
        0 & \text{otherwise}
    \end{cases}
\]
DFT of Hann window
Signal
Hann window
Windowed signal
DFT of signal
DFT of Hann window
DFT of windowed signal
DFT of windowed signal (rectangular window)
Time-frequency resolution

Time resolution governed by width of window

Can we just make the window arbitrarily narrow?
Compressing in time dilates in frequency and vice versa

$x \in L_2[-T/2, T/2]$ is nonzero in a band of width $2w$ around zero.

Let $y$ be such that

$$y(t) = x(\alpha t), \quad \text{for all } t \in [-T/2, T/2],$$

for some positive real number $\alpha$ such that $w/\alpha < T$.

The Fourier series coefficients of $y$ equal

$$\hat{y}[k] = \frac{1}{\alpha} \langle x, \phi_k/\alpha \rangle$$
Proof

\[ \hat{y} [k] = \int_{t=-T/2}^{T/2} y(t) \exp \left( -i \frac{2\pi kt}{T} \right) \, dt \]
Proof

\[ \hat{y} [k] = \int_{t=-T/2}^{T/2} y(t) \exp \left( -\frac{i2\pi kt}{T} \right) \, dt \]
\[ = \int_{t=-w/\alpha}^{w/\alpha} x(\alpha t) \exp \left( -\frac{i2\pi kt}{T} \right) \, dt \]
Proof

\[ \hat{y} [k] = \int_{t=\frac{-T}{2}}^{\frac{T}{2}} y(t) \exp \left( -\frac{i2\pi kt}{T} \right) \, dt \]

\[ = \int_{t=\frac{-w}{\alpha}}^{\frac{w}{\alpha}} x(\alpha t) \exp \left( -\frac{i2\pi kt}{T} \right) \, dt \]

\[ = \frac{1}{\alpha} \int_{\tau=\frac{-w}{\alpha}}^{\frac{w}{\alpha}} x(\tau) \exp \left( -\frac{i2\pi k\tau}{\alpha T} \right) \, d\tau \]
Proof

\[ \hat{y}[k] = \int_{t=-T/2}^{T/2} y(t) \exp \left( -\frac{i2\pi kt}{T} \right) dt \]
\[ = \int_{t=-w/\alpha}^{w/\alpha} x(\alpha t) \exp \left( -\frac{i2\pi kt}{T} \right) dt \]
\[ = \frac{1}{\alpha} \int_{\tau=-w}^{w} x(\tau) \exp \left( -\frac{i2\pi k\tau}{\alpha T} \right) d\tau \]
\[ = \frac{1}{\alpha} \int_{\tau=-T/2}^{T/2} x(\tau) \exp \left( -\frac{i2\pi k\tau}{\alpha T} \right) d\tau \]
Proof

\[ \hat{y}[k] = \int_{t = -T/2}^{T/2} y(t) \exp \left(-\frac{i2\pi kt}{T}\right) \, dt \]

\[ = \int_{t = -w/\alpha}^{w/\alpha} x(\alpha t) \exp \left(-\frac{i2\pi kt}{T}\right) \, dt \]

\[ = \frac{1}{\alpha} \int_{\tau = -w}^{w} x(\tau) \exp \left(-\frac{i2\pi k\tau}{\alpha T}\right) \, d\tau \]

\[ = \frac{1}{\alpha} \int_{\tau = -T/2}^{T/2} x(\tau) \exp \left(-\frac{i2\pi k\tau}{\alpha T}\right) \, d\tau \]

\[ = \frac{1}{\alpha} \langle x, \phi_{k/\alpha} \rangle \]
$w = 90$
$w = 30$
$w = 5$
Time-frequency resolution

Fundamental trade-off

Uncertainty principle: cannot resolve in time and frequency simultaneously
What have we learned

Effect of temporal windowing in the frequency domain

Trade-off in time-frequency resolution